UDC 519.21:519.61
Original scientific paper
Received: 20.02.2001.

A simple model for chaos studies in nonlinear feedback systems

Ahmet Uçar
Firat University, Engineering Faculty, Electrical - Electronics Eng., 23119 Elazig, TURKEY
e-mail: aucar1@firat.edu.tr

SUMMARY

Several models have been proposed in order to enlarge the class of nonlinear phenomena. In this article, a simple model is proposed for chaos studies in nonlinear feedback systems. The model consists of a state, time delay and a nonlinear element. It can be described as an autonomous continuous-time difference-differential equation with only one variable. The richness of such system behaviour is numerically illustrated and chaotic behaviour is presented. The computer generated chaos for the double-scroll attractor and bifurcation diagram are provided.

***Key words**: model for chaos, bifurcation diagram, time delay, nonlinear feedback systems.*

1. INTRODUCTION

Traditionally, it was implicitly assumed that random behaviour was due to extreme complexity of the dynamical systems with higher number of independent degrees of freedom. However, recent introduction of chaotic dynamics tells us that randomness in the dynamical systems does not necessarily involve an enormous number of independent degrees of freedom [1]. In the presence of a nonlinearity only a few independent variables are sufficient to generate chaotic motion. Consider the following deterministic ordinary differential equations:

$$\dot{x} = f(x), \qquad x(0) = x_0 \tag{1}$$

in which $x \in R^n$ is the state vector dependent on t and $\dot{x}$ denotes the derivative. The nonlinear function $f(.)$ may be dependent on t, x, and $f(.) : R_+ \times R^n \rightarrow R^n$. The initial state vector at $t=0$ is $x(0)$. The nonlinear function $f(.)$ may include a continuous or a discontinuous nonlinearity. For a possible chaotic behaviour, the system defined in Eq. (1) must contain at least three degrees of freedom, $n=3$. However, the nonlinearity and $n=3$ are not the only necessary conditions for a system to exhibit chaotic behaviour, but also the system trajectory has to be sensitive to initial conditions [1]. For the chaotic systems small changes in initial conditions are amplified into very large changes in the long-term behaviour, making the relationship between cause and effect so complicated as to be effectively unpredictable. This complexity of the behaviour is due to the internal, rather than external dynamics [2].

For autonomous continuous time nonlinear system it has been reported that chaos cannot occur when $n=1, 2$. Confirming this result, the engineering systems given in Ref. [3] have at least three state variables. Furthermore, models given to studying chaos such as Chua's circuits, Lorenz equations and pressure transducer model are all in third order forms [4-6]. This does not seem to be the case with systems with delay, a two-cell nonautonomous neural networks with delay may appear chaotic attractor [7]. Here further it will be shown that a simple continuous-time nonlinear system may exhibit chaotic behaviour without taking time delay too large.

2. THE SYSTEM DESCRIPTION

The engineering systems are nonlinear and generally modelled based on the assumption that the behaviour of the considered system depends on the present states only. Although this assumption is verified for large class of dynamical systems, there exist situations that the system's behaviour includes also information of former states. These systems are called time-delay systems. The study of the affects of time delay in such systems is not only a theoretical interest, but it has also a practical importance. The approximation of the time delay is not appropriate method in most situations and gives rise instability even in the linear systems. Furthermore, in most situations even system operated at nearby of an equilibrium point, time delay and nonlinearity both occur in the feedback systems and they could not be avoided such as time delay in sensors dynamics or controller structure [8]. Therefore, investigation of the dynamical behaviours of time delay system especially without any restriction is one of active research areas. Consider the following nonlinear continuous system in dimensionless form with one state variable:

$$\dot{x}(t) = d\, x(t-t) - e\, x^3(t-t) \qquad (2)$$

where d and e are the system parameters. Here t corresponds to the delay time in which it represents the time interval between the start of an event at one point and its resulting action at another point in the system. The time delay systems can be tackled from many points of view. In particular, the models of such systems can be considered as evolution in abstract systems, differential equations on rings or modulus, or as functional differential equations. However these approaches are limited to the linear systems and their extension methods to the nonlinear system are confined to illustrate the global behaviour of the nonlinear systems with delayed element.

Particularly the system given in Eq. (2) is used as a simple model to observe self-oscillations in the shipbuilding industry [9] and interestingly the complexity of the system has not been reported yet.

3. FROM REGULAR TO COMPLEX BEHAVIOURS

The effects of delay on both linear and nonlinear systems are studied and their stability is discussed in Ref. [10] with several approaches. Each described approach has some advantages or disadventages depending on the considered problem to be handled. The effect of time delay on the dynamical behaviour of one dimensional continuous nonlinear system is studied in Ref. [11] where describing function methods and pade approximation are employed for nonlinear parts and time delay, respectively. The magnitude and frequency of self-sustained oscillations are observed and their dependence on delay time is studied. It has been shown that the system given in Eq. (2) exhibits self-sustained oscillations with sufficiently short delay time. Increasing the time delay leads to undesirable phenomena and it results in instability.

In order to illustrate dynamics behaviour of the system let consider the system without time delay first i.e. $t=0$. In this case the system has three equilibrium points, namely origin and $\pm\sqrt{d/e}$. The vector field is illustrated in Figure 1. The arrows on the x-axis indicate the corresponding velocity vector at each interval of equilibrium points. The arrows point to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$. At equilibrium points where $\dot{x} = 0$, there is no flow.

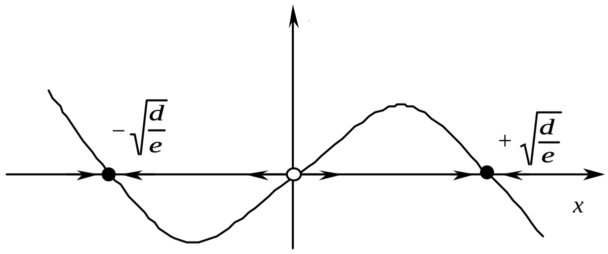

Fig. 1 Vector field of system given in Eq. (2) for $t=0$

In Figure 1 solid black dots at $\pm\sqrt{d/e}$ represent stable equilibrium points where open circle at origin represents the unstable one. Since such system behaviour is generally analyzed with small signal methods, the local behaviours of this equilibrium points are only obtained and discussed. Figure 1 shows that the system trajectory moves and approaches the stable fixed point at $\sqrt{d/e}$ for any initial condition satisfies $x_0>0$. Similarly, if a negative initial condition, $x_0<0$, is chosen then the system trajectory approaches to the negative equilibrium point at $-\sqrt{d/e}$ and it can be noted that there is no possibility of self-sustained oscillations in this case.

4. NUMERICAL RESULTS

The dynamical behaviours of the system described in Eq. (2) without any restriction may be only examined numerically. The system described in Eq. (2) is modelled in Matlab/SIMULINK® environment and it is numerically solved by the use of fifth order Runge-Kutta ordinary differential solver embedded in Matlab toolboxes [12]. The following numerical results are obtained for 0.001 integration step size and 10^{-6} absolute and relative tolerances. The delay time t and the nonlinearity gain e in Eq. (2) are fixed at unity. For $x(0)=0.1$ state initial condition, the system response is examined by change linear part gain d. Note that the delay time taken here is not too large and can occur in many systems especially due to the measurement devices. The system trajectory first is observed from time response and then the system trajectory is depicted in phase plane in such a manner that the integrator input is plotted versus its output as analogue of vector field illustrated in Figure 1.

The numerical results are obtained for $d=0.5$, 0.9, 1.51 and 1.7 and the system trajectories are depicted in phase plane, Figure 2 (a), (b), (c) and (d), respectively. Figure 2(a) shows the system phase portrait for $d=0.5$ and it indicates that the trajectory is asymptotically decaying at the positive equilibrium point since a positive initial condition is chosen. Conversely, for a negative initial condition the trajectory will decay onto the negative equilibrium point at $-\sqrt{d/e}$. Increasing the value of d to 0.9 leads to a self-oscillation as depicted in Figure 2(b) and encloses the stable equilibrium point at $\sqrt{d/e}$. One can show that this self-sustained oscillation is stable for a positive range of the state initial conditions. Note that the results illustrated in Figures 2 (a) and (b) are analogous to the results estimated by employing small signal analysis given in Ref. [11]. Choosing the parameter $d=1.5$ yields the behaviour of depicted in Figure 2(c). Here the system trajectory indicates several oscillations with different amplitude and frequencies. Note that the simulation interval time is chosen large enough for Figure 2(c) to observe both the transient and steady state responses of the system. The trajectories depicted in Figures 2(a)-(c) are usually considered as regular behaviours since their nature are known and can be observed analytically with approximation methods. However, increasing the values of d yields complex behaviours. For instance, choosing the parameter $d=1.7$ leads to the strange behaviour depicted in Figure 2(d).

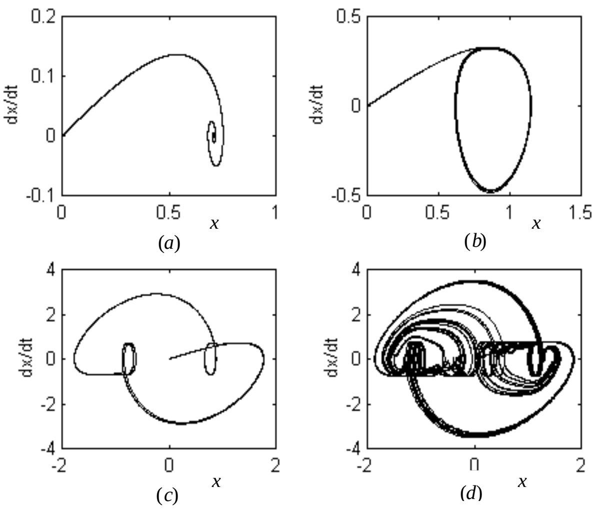

Fig. 2 Phase portrait of the system with $\tau=\varepsilon=1$: (a) for $d=0.5$, the system trajectory converges to the equilibrium point; (b) self-sustained oscillation for $d=0.9$; (c) oscillations for $d=1.51$; (d) chaotic behaviour for $d=1.7$

This strange behaviour can be no longer analyzed by methods given in Refs. [10, 11] since the trajectory depicted in Figure 2(d) has a broad band noise. There are no methods to allow one in order to separate noise-like solution from the output signal. The phase portrait depicted in Figure 2(d) is a clear indication of double-scroll chaotic type behaviour, observed in the study of chaotic systems given in Ref. [1] with three state variables.

The unique character of chaotic dynamics may be seen most clearly by examining the system to be started twice, but from slightly different initial conditions. For non-chaotic system, the uncertainty leads to an error in predicting that it grows linearly with time. However, for chaotic systems, on the other hand, the error grows exponentially in time, so that the state of the system is essentially unknown after a very short time. The time response of the system is obtained for two nearby initial conditions $x_0=0.1$ and $x_0=0.10001$ and a simultaneous difference taken between them, $e=x(t)_{0.1} - x(t)_{0.10001}$, is depicted in Figure 3 for $d=1.7$.

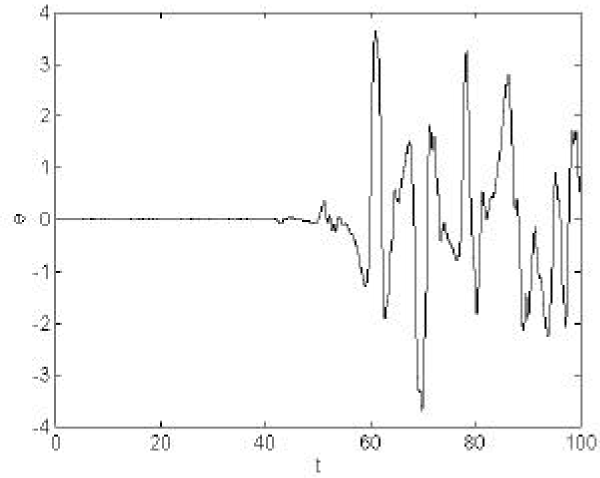

Fig. 3 The error, $e=x(t)_{0.1} - x(t)_{0.10001}$, between two trajectories obtained for nearby initial conditions of the system state

Figure 3 shows the error between two trajectories initially almost unchanged within the time interval of $0<t<40$, but it increases and gives raise to large error in later times resulting in a loss of final state predictability. The system output magnitude approximately changes between ± 2 as illustrated in Figure 2(d). However the error between system response of these two nearby initial conditions is larger than the actual signal and it varies between ± 4. This phenomenon is known as sensitivity of the system trajectory to the value of the initial conditions. The results depicted in Figure 2(d) and Figure 3 are a clear evidence of chaotic behaviour of the system in Eq. (2) with a reasonable time delay.

5. BIFURCATION DIAGRAM

The error between two system trajectories depicted in Figure 3 illustrates only the results for fixed system parameters. However, the details of the system responses to a range of a system parameter are considerably important. One of the potential uses is to avoid the chaotic regions by choosing appropriate parameter setting. It is also interesting in some situations, chaotic behaviours may needed. In order to obtain both regions two methods, namely, Bifurcations diagrams and Lyapunov exponent are introduced [12]. The former is resembling the system trajectories versus a system parameter in the periodically sampled Poincare section. The latter is monitoring the system trajectories versus system parameters and it indicates the sensitivity of the trajectory in terms of state initial condition [2].

Here the bifurcation diagram, which is a widely used technique for examining the pre-chaotic or post-chaotic changes in the system under parameter variations, is obtained. The system has three parameters: the gains of the linear and nonlinear parts and the delay time. For the fixed delay time the system behaviour qualitatively can be observed as a function of the system parameters ***d*** or *e*. However, it has been shown in Ref. [11] that by increasing the gain of the nonlinear part, ***e***, actually it stabilizes the system and does not effect the system behaviour qualitatively, since it negatively feeds the system. In order to obtain bifurcation diagram a Matlab program is developed based on the procedure given in Ref. [12]. The system parameter ***e*** is also fixed at unity and the system behaviour is observed by changing ***d*** within the range of $0.5 \leq d \leq 1.78$ with 0.2 step size. The system trajectories of the system are obtained for the last part of the simulation times of [0 20000] by eliminating possible transient responses.

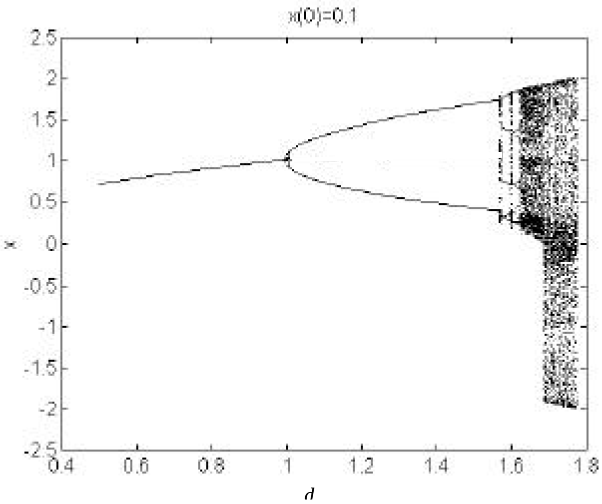

Fig. 4 Bifurcation diagram for the long term values of the system output x versus the parameter d

The system bifurcation diagram is depicted in Figure 4, in which the maximum system output x is plotted versus the selected range of parameters of ***d***. Figure 4 illustrates the system trajectories initially settled down at regular behaviours: a fixed point or a self-sustained oscillations for $d<1$. Increasing $d>1$ leads to period two until the value of d reaches 1.56 where a small chaotic region sets. For the range of $1.64<d<1.8$ the bifurcation diagram clearly indicates the chaotic region. Further increase of ***d*** leads to unbounded solutions, namely, unstable behaviour.

6. CONCLUSION

The dynamic behaviour of the simple continuous nonlinear system given in Eq. (2) with delay element is studied in this paper. It has been shown that the richness of the system can not be observed by the approximation methods. Interestingly, this system has been extensively used as a model for engineering systems and its dynamics has been studied, but the complexity of the system behaviour has not been mentioned. Here the chaotic behaviour of the system is shown and the system bifurcation diagram is obtained in order to illustrate the chaotic regions. The bifurcation diagram shows that some system parameters do not only lead to unstable response, but they also lead to chaotic behaviours, which cause practical problems. Another important result shows the limitation of input-output stability [14]. The effective results presented here may be considered as:

(1) A simple dynamical system with delay element can exhibit very complex behaviour including chaos.

(2) The system presented in this paper can be used as a prototype model for studying chaotic behaviours in general.

7. REFERENCES

[1] M.J. Ogorzalek, *Chaos and Complexity in Nonlinear Electronic Circuits*, Word Scientific, London, 1997.

[2] F.C. Moon, *Chaotic and Fractal Dynamics: An Introduction for Applied Scientists and Engineers*, John Wiley & Sons, New York, 1992.

[3] J.M.T. Thompson and S.R. Bishop, (Eds.), *Nonlinearity and Chaos in Engineering Dynamics*, John Wiley & Sons, Chichester, UK, pp. 1-21, 1994.

[4] M.P. Kennedy, Three step to chaos-part II: A choa's Circuit primer, *IEEE Trans. Circuits Syst.*, Vol. 40, pp. 657-674, 1993.

[5] C.T. Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractor*, Springer-Verlag, New York, 1982.

[6] C.P. Lewis, A. Uçar and S.R. Bishop, Dynamic behaviour of complex non-linear systems, Multi-body Dynamics: Monitoring and Simulation Techniques, (Eds. H. Rahnejat and R. Whalley), Mechanical Engineering Publ., London, pp. 201-208, 1997.

[7] M. Gilli, Strange attractors in delayed cellular neural networks, *IEEE Trans. Circuits Syst.*, Vol. 40, pp. 166-172, 1993.

[8] C.P. Lewis, A. Uçar and S.R. Bishop, Stability of nonlinear systems and the effects of time delay control, *Trans Inst. Measurement and Control*, Vol. 20, pp. 29-36, 1998.

[9] A.R.J. Lloyd, *Seakeeping: Ship Behaviour in Rough Weather*, Ellis Horwood Limited, Chichester, 1989.

[10] S.I. Niculescu, E.I. Verriest, L. Dugard and J.M. Dion, *Stability and Robust Stability of Time-Delay Systems: A Guided Tour, in Stability and Control of Time-Delay Systems* (Eds. L. Dugard and E.I. Verriest), Springer-Verlag, London, 1998.

[11] A. Uçar, Effects of delay on the dynamical behavior of one-dimensional continuous nonlinear systems, Proc. 6th IEEE Int. Conf. On Electronics, Circuits and Systems, ICECS'99, Vol. 2, pp. 933-936, 1999.

[12] D. Haselman and B. Littlefield, *Mastering Matlab 5: A Comprehensive Tutorial and Reference*, Prentice Hall, Upper Saddle River, New Jersey, 1998.

[13] T.S. Parker and L.O. Chua, *Practical Numerical Algorithms for Chaotic Systems*, Springer-Verlag, New York, 1989.

[14] C.P. Lewis, A. Uçar and S.R. Bishop, An approach to the investigation of robustness and stability of delay feedback control of nonlinear systems, Proc. 15th IMACS World Congress on Scientific Computation, Modelling and Applied Mathematics, (Ed. A. Sydow), Wissenschaft and Verlag, Vol. 5, pp. 263-268, 1997.

JEDNOSTAVAN MODEL ZA PROUÈAVANJE KAOSA KOD NELINEARNIH SUSTAVA S POVRATNOM VEZOM

SAŽETAK

Predlaže se nekoliko modela za poveæanje skupina nelinearnih fenomena. U ovom se èlanku predlaže jedan jednostavan model za prouèavanje nelinearnih sustava s povratnom vezom. Model se sastoji od stanja, odgode vremena i nelinearnog elementa. Može se opisati kao autonomna stalna diferencijalna jednadžba razlike vremena koja ima samo jednu varijablu. Složenost takvog ponašanja sustava opisano je numerièki èime je moguæe prikazati i kaotièno ponašanje. Time je omoguæen raèunalom generiran kaos za dvostruko klizajuæi atraktor i dijagram bifurkacije.

***Kljuène rijeèi**: model za kaos, dijagram bifurkacije, odgoda vremena, nelinearni sustavi s povratnom vezom.*