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Strain field in doubly curved surface

Amod Tiwari(1), **Aurobinda Chatterjee**(1), **Vinay K. Pathak**(2) and **Sanjay G. Dhande**(3)
(1)CAD Laboratory, Indian Institute of Technology Kanpur, Kanpur 208 016, INDIA
e-mails: amod@iitk.ac.in; achat@iitk.ac.in
(2)Harcourt Butler Technological Institute Kanpur, Nawabganj, Kanpur-21, INDIA
e-mail: vpathak@lycos.com
(3)Indian Institute of Technology Kanpur, Kanpur 208 016, INDIA
e-mail: sgd@iitk.ac.in

SUMMARY

This paper presents algorithm for development of structural and continuous curved surface into a planar and non planar (radial) shape in 3D space. The development process is modeled by application of strain in certain plane from the curved surface to its planar development. A doubly curved surface has been generated for the purpose of technical studies. Important features of the approach include formulations of the coefficients of first fundamental form, second fundamental form, Gaussian curvature and Serret Frenet curve. The approximate strain field is obtained by solving a constrained linear and nonlinear problem in algorithm.

Key words: *doubly curved surface, fundamental forms, Frenet curve radius, Gaussian curvature, strain field, Serret angle.*

1. INTRODUCTION

Shape design and representation of complex objects such as car, ship, airplane, etc., could not be achieved through wireframe modeling. These objects are better modeled through surface modeling. Generation of surface in general requires some quantitative data, such as a set of points and tangents as well as some qualitative data, such as intuition for the desired shape and smoothness. The choice of the surface geometry depends upon the application and the manufacturing methods needed to produce the surface. Polynomial functions are easy to deal with [1]. The higher order polynomials are avoided for surface design problems due to large number of coefficients, which make it difficult to control the resulting surface. For most practical surface applications, cubic polynomials are sufficient.

Two kinds of surfaces, developable surfaces and non-developable surfaces are used in engineering applications [2, 3]. These are also called as ‘singly’ and ‘doubly curved’ surfaces respectively. A developable surface has zero values of Gaussian curvature at all points while a non-developable surface has non-zero values of Gaussian curvature at least in some region [4].

A developable surface is highly favorable in metal forming since it can be formed only by bending without any tearing and stretching. However surfaces of many engineering structures are commonly fabricated as doubly curved shapes to fulfill the functional requirements such as hydrodynamic, aesthetic or structural.

For example, a large portion of shell plates of ship hulls or airplane fuselages and prosthetic sockets are doubly curved surfaces [5]. Given a three-dimensional design surface, which represents a face of a curved plate or shell, the first step in the fabrication process is flattening or planar development of this surface into planar shape. So that the manufacturer can determine the initial shape of the flat plate, also many estimate the strain distribution required to obtain the desired shape. A planar development corresponding to minimum stretching or shrinkage is highly desirable because it saves material and it reduces the work required. Early surface development procedures were implemented in shipyards based on geodesic development during the last three decades, mainly for ship hull plates, whose Gaussian curvature is very small [4]. Manning [6] has developed a procedure for surface development based on an isometric tree. A tree of lines with a spine and branches was first drawn on the curved surface. Then these were developed isometrically onto planar curves, using the geodesic curvature of the spine and branches on the surface as the curvature of the planar curves. The envelope of the developed pattern formed the planar developed shape. The shape of planar development is dependent on the choice of the spine and branch curves. The stretching along both spine and branch curves was zero. Hinds [7] has developed patterns for simple surfaces that have been considered in terms of Gaussian curvature. Patterns are derived for elliptic and hyperbolic curvature regions. Similar methods have been applied to garment pieces that have been defined as 3D mathematical models at a workstation. Approximations have been applied to patterns to reduce the complexity to a level that is acceptable to the clothing industry. The limitation of the method is that the developed shape depends on the choice of starting edge. If used in metal forming, it is not guaranteed that the forming process would be realizable from the planar shape to the curved shape.

2. LITERATURE SURVEY

Letcher [8] has presented a basic geometric theory for flattening and fabrication of doubly curved plates. The mapping from the curved surface to its planar development has been obtained by adding in-plane strain to the curved surface. The strain field has been obtained by solving a generalized Poisson's equation with the source term equal to the Gaussian curvature. Cho [9] has presented an algorithm to approximately develop a doubly curved surface by minimizing the mapping error function for locally isometric mapping between a given and developed surface net. The method has been applied to construct an auxiliary planar domain of triangulation for tessellating trimmed parametric surface patches, which sufficiently preserves the shape of triangles when mapped into three-dimensional space. Cho's method for metal forming has some inherent problems. Azariadis [10] have dealt with the approximate design of planar development of doubly curved surfaces and their refinement in order to derive a final pattern with limited gaps and overlaps. They have divided the problem of planar development into three stages i.e.:

- Stage of defining the starting guide strip,
- Stage of designing the initial pattern, and
- Stage of refinement.

The method dealing with first stage of problem is based on elements of geodesic and Gaussian curvature. They have also presented an alternative technique of generating an initial planar development of doubly curved surface. Yu [4] has presented algorithms for optimal development (flattening) of a smooth continuous surface embedded in three-dimensional space into planar shape.

The development process has been modeled by in-plane strain (stretching) from the curved surface to its planar development. The distribution of the appropriate minimum strain field has been obtained by solving a constrained nonlinear programming problem. Based on strain distribution and coefficients of the first fundamental form of curved surface, another unconstrained nonlinear programming problem has been solved to obtain the optimal developed planar shape. Michael [5] have presented an optimal approach to laser scanning paths and heating condition determination for laser forming of doubly curved shapes. Important feature in their approach includes strain field calculation based on the Serret Frenet curvature formulations and minimal strain optimization, and scanning paths and heating condition determined by combining analytical and practical constraints.

Different curved surfaces have been shown in Figure 1 considering single, double and triple curve states.

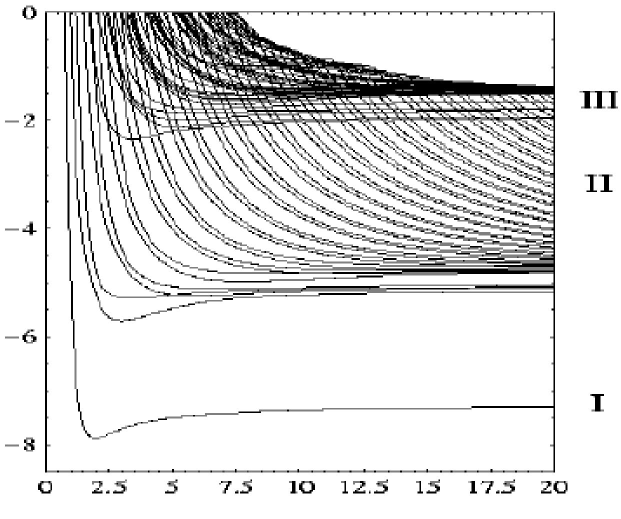

Fig. 1 Curved surfaces: I - Support single curve state; II - Support double curve state; III - Support triple curve state

In the present work, an attempt has been made for optimal development of a doubly curved surface such that the strains from the surface to its planar development are minimized. The development from curved surface to planar shape is modeled by in-plane strain methodology. An example has been solved, where the surface has both positive and negative values of Gaussian curvature.

3. GENERALIZATION OF DOUBLY CURVED SURFACE

An arbitrary surface is not defined globally, but only at some local points. As a result, the local shape can be known and be determined by the derivatives of the surface vector. The nature of local shape is classified into four cases according to the Gaussian curvature K_1, K_2 and the coefficients of the second fundamental form viz. L, M and N at a point. The commonly used surfaces are:

(i) ***elliptic surface***, which has positive Gaussian curvature. In Figure 2, the inter section of K_1 and K_2 curves denote the positive Gaussian curvature (peak representation).

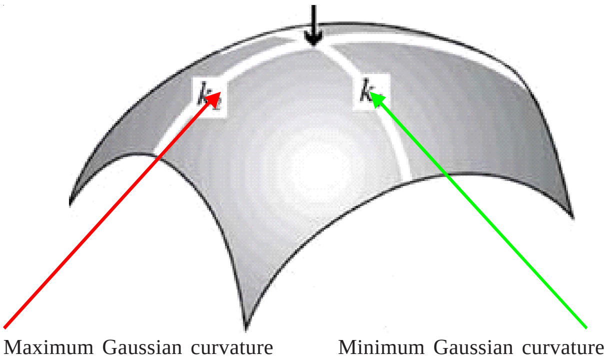

Fig. 2 Elliptical doubly curved surface

(ii) ***hyperbolic surface***, which has negative Gaussian curvature. At a point where surface is neither concave nor convex but a surface with a radius of curvature tending towards infinity is known as saddle point, Figure 3.

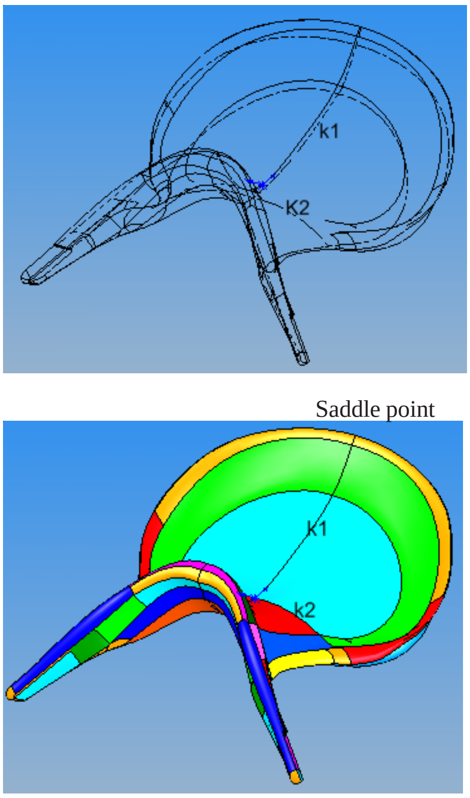

Fig. 3 (a, b) Hyperbolic doubly curved surface of saddle

(iii) ***parabolic surface***, which has zero Gaussian curvature, Figure 4.

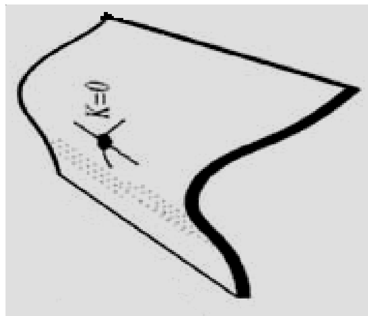

Fig. 4 Parabolic doubly curved surface

(iv) ***planar surface***, which has zero Gaussian curvature, Figure 5 and $L=M=N=0$.

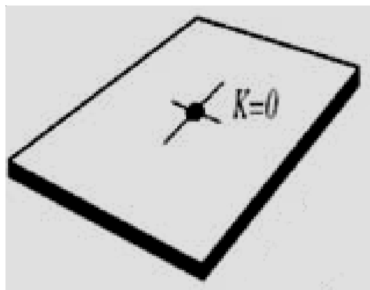

Fig. 5 Plane doubly curved surface

The conditions in each case are to be satisfied at all major points. A doubly curved surface, Figure 5, generally requires both in-plane and bending strains to form. For thin plates, in-plane strain is usually much larger than the bending strain and therefore only the former is considered. A doubly curved surface has been generated for the purpose of study in this work. The surface has a form of Bezier surface as shown in Figure 6.

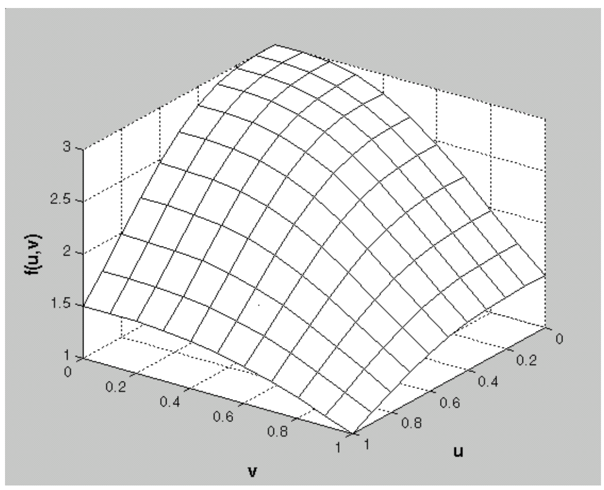

Fig. 6 The generated shape

Points on the Bezier surface are given by the following equation [1]:

$$P(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}P_{ij}B_{i,n}(u)B_{j,m}(v) \qquad (1)$$

where $u,v \in [0,1]$, P_{ij} is the ij^{th} control point. These points form the vertices of the control or characteristic polyhedron of the resulting Bezier surface; $B_{i,n}(u)=C(n,i)\, u^i(1-u)^{n-i}$ where $C(n,i)$ is the binomial coefficient and is given as:

$$C(n,i)=\frac{n!}{i!\times(n-i)!}$$

In the design environment, the Bezier surface is superior to bi-cubic surface because it does not require tangent or twist vectors to define a surface. Its main disadvantage is the lack of local control. By changing one or more control points affects the shape of whole surface. The user cannot change the shape of part of the surface also. Therefore a bi-cubic Bezier surface has been selected to have more control over the surface [1]. For a bi-cubic Bezier surface patch, sixteen points are required to determine the surface patch [5]. The point on the surface patch can be expressed in the matrix form [1]:

$$\boldsymbol{P}(u,v)=\left[(1-u)^3\; 3u(1-u)^2\; 3u^2(1-u)\, u^3\right]\times\boldsymbol{Q}\times\begin{bmatrix}(1-v)^3\\ 3v(1-v)^2\\ 3v^2(1-v)\\ v^3\end{bmatrix} \qquad (2)$$

where $\boldsymbol{Q}$ is a matrix having $(m+1)$ rows and $(n+1)$ columns containing sixteen control points.

The following control points for the surface generation have been taken:

{(0,0,3), (0,1/3,2.89), (0,2/3,2.31), (0,1,1.5)},
{(1/3,0,2.89), (1/3,1/3,2.79), (1/3,2/3,2.25), (1/3,1,1.47)},
{(2/3,0,2.31), (2/3,1/3,2.25), (2/3,2/3,1.88), (2/3,1,1.31)},
{(1,0,1.5), (1,1/3,1.47), (1,2/3,1.31), (1,1,1)}.

3.1 Surface representation

The function $\boldsymbol{P}(u,v)$ at certain values u and v is the point on the surface at these values. The general way to describe the parametric equation of a three-dimensional curved surface in space is:

$$\boldsymbol{P}(u,v)=[x\ y\ z]^T=[x(u,v)\ y(u,v)\ z(u,v)]^T \qquad (3)$$

where $u_{min}\le u \le u_{max}$, $v_{min}\le v \le v_{max}$

The above equation suggests that a general three-dimensional surface can be modeled by dividing it into an assembly of topological patches. A patch is considered the basic mathematical element to model a complete surface. Some surface may consist of one patch only while others may be a few patches connected together. The topology of the patch may be rectangular or triangular. There are two types of surfaces, analytic and synthetic. Analytic surfaces are based on wireframe entities and synthetic surfaces are formed from a given set of data points or curves. The tensor product method is the most popular and is widely used in surface modeling. The tensor product formulation is a mapping of a rectangular domain described by the u and v values. Tensor product surfaces fit naturally onto rectangular patches [11]:

$$u_j^i=\begin{cases}1, & \text{if } i=j\\ 0, & \text{if } i\neq j\end{cases} \qquad \begin{matrix}\textit{For first domain}\\ \textit{(by definition of Kronecker delta)}\end{matrix}$$

$$v_j^i=\begin{cases}1, & \text{if } i=j\\ 0, & \text{if } i\neq j\end{cases} \qquad \begin{matrix}\textit{For second domain}\\ \textit{(by definition of Kronecker delta)}\end{matrix}$$

There is a set of boundary conditions associated with a rectangular patch. There are sixteen vectors and four boundary curves, shown in Figure 7. The vectors are four position vectors for the four corner points, eight tangent vectors and four twist vectors at the corner points. Geometric surface analysis is performed using principles of differential geometry. The parametric surface $\boldsymbol{P}(u,v)$ is directly amenable to differential analysis. There are intrinsic differential characteristics of a surface such as the unit normal and the Serret Frenet curvatures and directions which are independent of parameterization.

4. FIRST FUNDAMENTAL FORM

The tangent vector at any point $\boldsymbol{P}(u,v)$ on the surface is obtained by keeping one parameter as a constant and differentiating position vector of the point with respect to the other.

These vectors are given by:

$$\boldsymbol{P}_u(u,v)=\delta\boldsymbol{P}/\delta u=\delta x/\delta u\ \boldsymbol{i}+\delta y/\delta u\ \boldsymbol{j}+\delta z/\delta u\ \boldsymbol{k} \qquad (4)$$

where $u_{min}\le u\le u_{max}$, $v_{min}\le v\le v_{max}$ along the v=constant curve, and:

$$\boldsymbol{P}_v(u,v)=\delta\boldsymbol{P}/\delta v=\delta x/\delta v\ \boldsymbol{i}+\delta y/\delta v\ \boldsymbol{j}+\delta z/\delta v\ \boldsymbol{k} \qquad (5)$$

where $u_{min}\le u\le u_{max}$, $v_{min}\le v\le v_{max}$ along the u=constant curve. Tangent vectors are useful in determining boundary conditions for patching surfaces together. The magnitudes and unit vectors of the tangent vectors are given by:

$$|\boldsymbol{P}_u|=\sqrt{\left\{(\partial x/\partial u)^2+(\partial y/\partial u)^2+(\partial z/\partial u)^2\right\}} \qquad (6)$$

$$|\boldsymbol{P}_v|=\sqrt{\left\{(\partial x/\partial v)^2+(\partial y/\partial v)^2+(\partial z/\partial v)^2\right\}} \qquad (7)$$

$$\boldsymbol{n}_u=\boldsymbol{P}_u/|\boldsymbol{P}_u| \quad \boldsymbol{n}_v=\boldsymbol{P}_v/|\boldsymbol{P}_v| \qquad (8)$$

Twist vector at a point is the rate of change of the tangent vector $\boldsymbol{P}_u$ with respect to v or vice-versa. The twist vector depends upon both the surface geometric characteristics and its parametrization. The twist vector can be written in terms of its Cartesian components as:

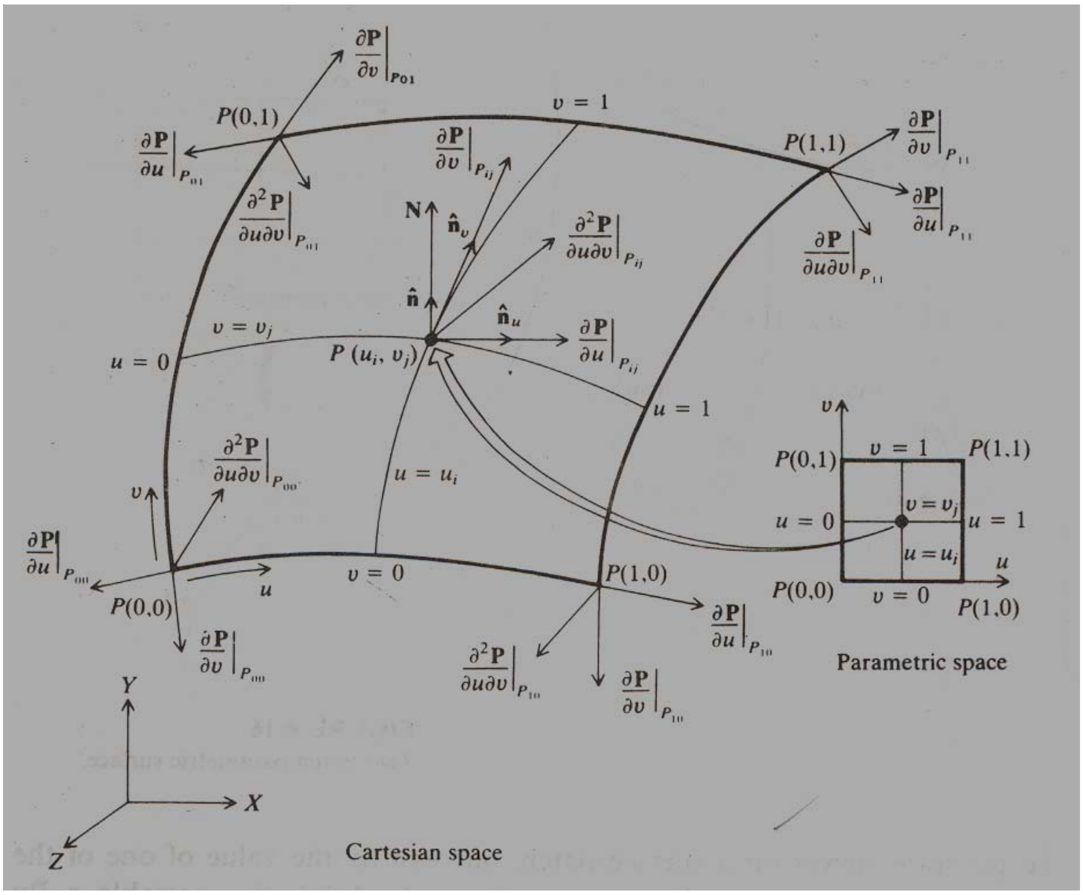

Fig. 7 A parametric surface patch with its boundary conditions

$$\boldsymbol{P}_{uv}=\left[\partial^2 x/\partial u\partial y \quad \partial^2 y/\partial u\partial y \quad \partial^2 z/\partial u\partial y\right]^T =$$
$$= \partial^2 x/\partial u\partial y\, \boldsymbol{i} + \partial^2 y/\partial u\partial y\, \boldsymbol{j} + \partial^2 z/\partial u\partial y\, \boldsymbol{k} \qquad (9)$$

where $u_{min} \leq u \leq u_{max}$, $v_{min} \leq v \leq v_{max}$.

The normal to the surface is another important analytical property. The surface normal at a point is a vector, which is perpendicular to both tangent vectors at the point, that is:

$$\boldsymbol{N}(u,v)=(\delta\boldsymbol{P}/\delta u)\times(\delta\boldsymbol{P}/\delta v)=\boldsymbol{P}_u\times\boldsymbol{P}_v \qquad (10)$$

and the unit normal vector is given by:

$$\boldsymbol{n}=\boldsymbol{N}/|\boldsymbol{N}|=(\boldsymbol{P}_u\times\boldsymbol{P}_v)/|\boldsymbol{P}_u\times\boldsymbol{P}_v| \qquad (11)$$

The calculation of the distance between two points on a curved surface becomes an important aspect for surface analysis. The infinitesimal distance between two points (u,v) and $(u+du,\ v+dv)$ on a surface is given by:

$$ds^2 = \boldsymbol{P}u\cdot\boldsymbol{P}u\ du^2+2\boldsymbol{P}u\cdot\boldsymbol{P}v\ du\ dv+\boldsymbol{P}v\cdot\boldsymbol{P}v\ dv^2 \quad (12)$$

Equation (12) is called as the first fundamental quadratic form of the surface and is written as:

$$ds^2=E\ du^2+2F\ du\ dv+G\ dv^2 \qquad (13)$$

where: $E(u,v) = \boldsymbol{P}_u\cdot\boldsymbol{P}_u$
$F(u,v) = \boldsymbol{P}_u\cdot\boldsymbol{P}_v$
$G(u,v) = \boldsymbol{P}_v\cdot\boldsymbol{P}_v$

E, F and G are the first fundamental or metric coefficients of the surface. These coefficients provide the basis for the measurement of length and areas, and the specification of directions and angles on a surface.

4.1 Second fundamental form

The first fundamental form yields no information on how surface curves away from the tangent plane at that point. To investigate the surface curvature, another distance perpendicular to the tangent plane at $P(u,v)$ is introduced and is given by:

$$\tfrac{1}{2}\, dh^2 = \boldsymbol{n}\cdot\boldsymbol{P}_{uu}\ du^2 + 2\boldsymbol{n}\cdot\boldsymbol{P}_{uv}\ du\ dv + \boldsymbol{n}\cdot\boldsymbol{P}_{vv}\ dv^2 \quad (14)$$

The above equation is often called the second fundamental quadratic form of the surface and is given as:

$$\tfrac{1}{2}\, dh^2 = L\ du^2 + 2M\ du\ dv + N\ dv^2 \qquad (15)$$

where $L(u,v)=\boldsymbol{n}\cdot\boldsymbol{P}_{uu}$, $M(u,v)=\boldsymbol{n}\cdot\boldsymbol{P}_{uv}$, $N(u,v)=\boldsymbol{n}\cdot\boldsymbol{P}_{vv}$
L, M and N are the second fundamental coefficients, L_1, M_1, N_1 and L_2, M_2, N_2 are direction cosine of the plane surface and form the basis for defining and analyzing the curvature of a surface:

$$\cos(\theta)=\frac{L\,L_1 + M\,M_1 + N\,N_1}{\sqrt{\left(L_1^2+M_1^2+N_1^2\right)}*\sqrt{\left(L_2^2+M_2^2+N_2^2\right)}} \qquad (16)$$

$$\cos(\Phi)=\frac{L\,L_2 + M\,M_2 + N\,N_2}{\sqrt{\left(L^2+M^2+N^2\right)}*\sqrt{\left(L_2^2+M_2^2+N_2^2\right)}} \qquad (17)$$

4.2 Gaussian curvature

The surface curvature at the point on a normal section curve given by the form $\{u=u(t),\ v=v(t)\}$ can be written as [4]:

$$\kappa_n = \frac{L\,u'^2 + 2M\,u'v' + N\,v'^2}{E\,u'^2 + 2F\,u'v' + G\,v'^2} \tag{18}$$

The radius of curvature at the point is $r=1/\kappa_n$.

The above equation gives the surface curvature in any direction at point $P(u,v)$. During analysis of curved surface two curvatures viz. Gaussian curvature and mean curvature are used. The Gaussian curvature (K) and mean curvature (H) are defined by Michael [5], Zeid [1] and Carmo [12] as:

$$K = \frac{(L\times N)}{(E\times G) - F^2} \tag{19}$$

$$H = \frac{(E\times N)+(G\times L)-(2\times F\times M)}{2\times\left((E\times G)-F^2\right)} \tag{20}$$

4.3 Serret Frenet curvature

The values of Gaussian curvature and mean curvature are used to obtain the Serret Frenet curvatures [13], which are the upper (maximum) and lower (minimum) bounds on the curvature at the point [1]:

$$t = \frac{dr}{ds} \tag{21}$$

$$\frac{dt}{ds} = kn \tag{22}$$

$$\frac{dn}{ds} = -kt + \tau b \tag{23}$$

$$\frac{db}{ds} = -\tau n \tag{24}$$

It is clear that t, n and b are mutually perpendicular to each other, Figure 8. If we consider t, b and n are vectors quantity, therefore:

$$b = t\times n \tag{i}$$

$$t = n\times b \tag{ii}$$

$$n = b\times t \tag{iii}$$

Now, we can find the relation between Gaussian curvature (K), mean curvature (H) and Serret Frenet constant (κ).

The curvature constant (κ) has been defined two types from Eqs. (25) and (26):

(i) κ_{max} is maximum geodesic distance between doubly curve surface, and

(ii) κ_{min} is minimum geodesic distance between doubly curve surface, see Figure 3, as:

$$\kappa_{max} = H + \sqrt{\left(H^2 - K\right)} \tag{25}$$

$$\kappa_{min} = H - \sqrt{\left(H^2 - K\right)} \tag{26}$$

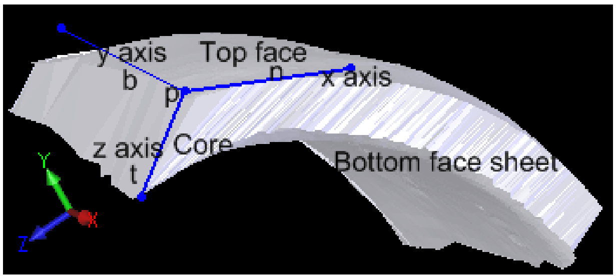

Fig. 8 Doubly curved surface

5. DETERMINATION OF STRAIN FIELD

During metal forming by line heating [14], the strains due to development from curved surface to its planar development are:

$$\varepsilon^u(u,v) \geq 0$$

and:

$$\varepsilon^v(u,v) \geq 0.$$

Normal strains are a non-dimensional quantity defined by the ratio of extension or shrinkage of a fiber and its original length as shown in Figure 9. After surface development, an infinitesimal length $|\boldsymbol{P}_u du|$ changes to $(1+\varepsilon^u)|\boldsymbol{P}_u du|$ and an infinitesimal length $|\boldsymbol{P}_v dv|$ changes to $(1+\varepsilon^v)|\boldsymbol{P}_v dv|$.

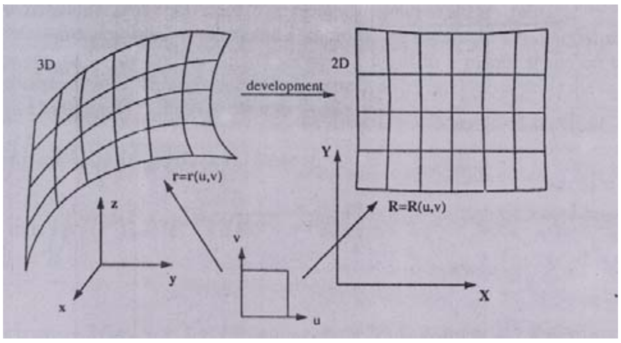

Fig. 9 Curced surface and its planar developent

Thus we have:

$$|\boldsymbol{p}_u| = (1+\varepsilon^u)|\boldsymbol{P}_u|,\quad |\boldsymbol{p}_v| = (1+\varepsilon^v)|\boldsymbol{P}_v| \tag{27}$$

where $\boldsymbol{p}(u,v)$ is the planar development. The first fundamental form coefficients of the developed surface $\boldsymbol{p}(u,v)$ are given by Yu [4]:

$$e = \boldsymbol{p}_u\cdot\boldsymbol{p}_u$$
$$f = \boldsymbol{p}_u\cdot\boldsymbol{p}_v \tag{28}$$
$$g = \boldsymbol{p}_v\cdot\boldsymbol{p}_v$$

After substituting Eq. (27) and the relations:

$$\boldsymbol{p}_u\cdot\boldsymbol{p}_u = |\boldsymbol{p}_u|^2,\qquad \boldsymbol{p}_v\cdot\boldsymbol{p}_v = |\boldsymbol{p}_u|^2$$

into Eq. (28), the coefficients of the first fundamental form of the planar developed surface are [4]:

$$e = (1+\varepsilon^u)^2\, E$$
$$f = (1+\varepsilon^u)\cdot(1+\varepsilon^v)\, F \qquad (29)$$
$$g = (1+\varepsilon^v)^2\, G$$

It is therefore required to minimize the strains $e^u(u,v)$ and $e^v(u,v)$, which satisfy the condition that after adding these strains to the doubly curved surface, it maps to a planar shape on which the Gaussian curvature is zero. This minimization is done in an integral sense using the squares of the strains.

This results are presented as follows [4]:

$$\min \iint \left\{\left(\varepsilon^u\right)^2 + \left(\varepsilon^v\right)^2\right\} \left|\boldsymbol{P}_u \times \boldsymbol{P}_v\right| du\, dv =$$
$$= \min \iint \left\{\left(\varepsilon^u\right)^2 + \left(\varepsilon^v\right)^2\right\} \sqrt{\left(EG - F^2\right)}\, du\, dv \qquad (30)$$

Such that:

$$0 = \{e\,(e_v g_v - 2 f_u g_v + g_u{}^2) +$$
$$+ f\,(e_u g_v - e_v g_u - 2e_v f_v + 4e_u f_v - 2 f_u g_u) +$$
$$+ g\,(e_u g_u - 2e_u f_v + e_v{}^2) -$$
$$- 2(eg - f^2)(e_{vv} - 2 f_{uv} + g_{uu})\}/4(eg - f^2)^2 \qquad (31)$$

where: $\varepsilon^u(u,v) \geq 0$,
$\varepsilon^v(u,v) \geq 0$, and:
$(u,v) \in D$.

This constrained minimization problem is discretized by using the finite difference method and trapezoidal rule of integration [15]. A grid of $(N_g{}^u \times N_g{}^v)$ points in the parametric domain is used in the discretization. Therefore, the total numbers of variables are $(2\times N_g{}^u \times N_g{}^v)$. To guarantee the independence of each constraint, constraints are imposed at the internal points of the grid, so there are $[(N_g{}^u-2)\times(N_g{}^v-2)]$ constraints. After discretization, the objective function becomes [4]:

$$\sum_{i=1}^{N_g{}^u} \sum_{j=1}^{N_g{}^v} \alpha_{ij} \left\{\left(\varepsilon_{ij}{}^u\right)^2 + \left(\varepsilon_{ij}{}^v\right)^2\right\} \sqrt{\left(E_{ij} G_{ij} - F_{ij}{}^2\right)}\, \Delta u\, \Delta v \qquad (32)$$

where following the trapezoidal rule of integration [10]:

$$\alpha_{ij} = 1 \text{ when } 1<i<N_g{}^u;\ 1<j<N_g{}^v,$$
$$\alpha_{ij} = 0.5 \text{ when } 1<i<N_g{}^u;\ j=1 \text{ or } j=N_g{}^v,$$
$$\alpha_{ij} = 0.5 \text{ when } i=1 \text{ or } i=N_g{}^u;\ 1<j<N_g{}^v, \qquad (33)$$
$$\alpha_{ij} = 0.25 \text{ when } i=j=1 \text{ or } i=N_g{}^v, j=N_g{}^v,$$
$$\alpha_{ij} = 0.25 \text{ when } i=N_g{}^u, j=1 \text{ or } i=1, j=N_g{}^v,$$

We use second order central difference methods to approximate all the derivatives in Eq. (31) at the internal points of the grid. After discretization, we obtain a nonlinear optimization problem with a convex cost function and nonlinear polynomial constraints. This nonlinear programming problem is solved using the *Matlab routine fmincon*, which is designed to solve the nonlinear programming problem. In implementation of this work, the starting point of the minimization is that all the strains are chosen to be zero.

6. RESULT

In the present work, the entire surface has been divided into *25* grid points. For the purpose of studies, number of grid points has been taken as *5* (that is $N_g{}^u=5$ and $N_g{}^v=5$). The strain distribution after the constrained minimization problem was solved using tolerances of 10^{-2} for the constraints and 10^{-3} for the objective function. The strains are scaled to fit into the figure (Figure 10).

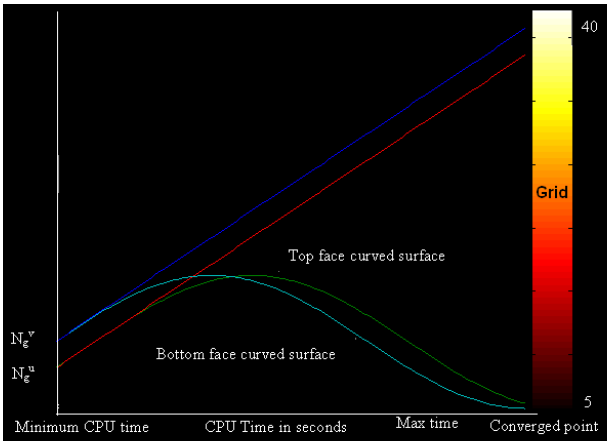

Fig. 10 Graph between Grid values ($N_g{}^u$, $N_g{}^v$) and CPU time

The extreme values of the strain field are located at $(u,v)=(0.5, 0.25)$ with $(\varepsilon^u,\varepsilon^v)=(0.023591, 0.389654)$. The objective function converges to the value of 2.56814×10^{-3} at the solution. All the constraints have been found within the tolerance of 1×10^{-2}. Table 1 shows the CPU time spent in optimization for various numbers of grid points and the values of objective functions. In this table N_g represents the number of grid points in both *u* and *v* directions, *Niter* is the number of iterations, *obj* is the converged value of the objective function in the optimization process and *CPU* time is time spent on the optimization.

Table 1 CPU time for each optimization at various number of grid points

Grid Points (N_g)	Niter	Obj (10^{-3})	CPU Time (seconds)
5	3	2.56814	0.61
7	8	2.49626	8.53
10	7	2.46140	33.00
15	13	2.44710	366.71
20	33	2.39968	7862.68
25	40	2.38560	13210.65
30	52	2.36140	20100.20
40	57	2.35110	29265.40

7. CONCLUSIONS

The value of strain is greater than compared to the surfaces having positive or negative values of Gaussian curvature only. The number of grid points should be kept at around 30 because any further increase in number of grid points does not significantly decrease the objective function value but increases number of iterations as well as CPU time. As the number of grid points is increased, the CPU time, which is time taken to optimize the objective function increases drastically. The values of the objective function and the constraints are within the tolerance limit. The values of the constraints will never be zero.

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POLJE DEFORMACIJA NA DVOSTRUKO ZAKRIVLJENOJ PLOHI

SAŽETAK

U ovom radu predstavlja se algoritam za razvoj konstrukcijske i kontinuirane zakrivljene plohe u ravni i zakrivljeni (radijalni) oblik u 3-D prostoru. Razvojni proces je modeliran primjenom deformacije na određenoj plohi od zakrivljene plohe do ravne. Dvostruko zakrivljena ploha proizvedena je zbog tehničkog proučavanja. Važne značajke pristupa uključuju formulaciju koeficijenta prvog osnovnog oblika, Gaussove zakrivljenosti i Serret Frenetove krivulje. Aproksimacijsko polje deformacija dobije se rješavanjem neprirodnog linearnog i nelinearnog problema u algoritmu.

Ključne riječi: *dvostruko zakrivljena ploha, osnovni oblici, radius Frenetove krivulje, Gaussova zakrivljenost, deformirano polje, Serretov kut.*