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Fractals and formfinding - magic with real numbers

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SUMMARY

The equilibrium shape determination of a prestressed cable net structures termed "formfinding" can be achieved through the classical Newton-Raphson iterative method. The convergence strongly depends on the choice of the starting aproximation vector, which must lie within the "ball of convergence" or "basin of attraction" unknown in advance. From experience this basin of attraction is very irregular, hence it was hypothesized that it may be a fractal. For an explanation of this irregularity, a series of numerical experiments was performed with highly simplified cable nets. It was confirmed that basins of attraction were fractals indeed. Cross-sections of these three dimensional fractals are visualized, and they appear highly interesting and diverse. It is shown that the Newton-Raphson method can also generate fractals with real numbers. Experiments were also performed with other iterative algorithms, with similarly interesting results - both from the equation solving and from the fractal producing points of view. It is believed that all discussed methods (except for the force density method which is specific to the formfinding problem) can be also applied to explain convergence properties of many other nonlinear equation systems.

***Key words**: cable net structures, formfinding, basin of attraction, fractals, iterative methods, nonlinear equations.*

1. INTRODUCTION

1.1 Mathematical fractals

The "mathematical" fractals (fractals generated by some mathematical recursive or iterative algorithm, in contrast to materials like clouds, coastlines, lungs and other known and unknown examples) can be generally divided into two classes, which could be termed: "synthetic" (recursive) and "analytic" (iterative). In broad terms, whereas synthetic fractals are invented (Koch, Sierpinski ...), analytic ones are discovered (Mandelbrot, Julia, Newton-Raphson on the complex plane ...) [1, 2]. More or less, all known "analytic" fractals are generated by nonlinear algorithms with complex (or quaternion) arithmetic. Therefore it is not surprising that among some members of the fractal community there exists an unstated belief in almost magic properties of complex numbers compared to real numbers. Here are some characteristic citations:

'*So a fractal is essentially a graph of an iterative process applied to complex numbers.*'

'*Fractals formed from real numbers are pretty, but you should see what complex numbers can do ...*'

'*Secondly, and paradoxically, its border is infinite. This is because Mandelbrot's equation dealt in complex numbers rather than real numbers.*'

Here it will be argued that complex arithmetic is not an essential condition for the generation of "analytic" fractals. Reference to fractals and real numbers was alluded to in Ref. [3] where a complex equation has been separated into a real and an imaginary part and iteration was performed for the resulting system of two real equations. In the reference

[4] generalization of the Mandelbrot algorithm in the real domain is described.

Fractals described in this paper were discovered while trying to explain strongly irregular behaviour of Newton-Raphson iteration for systems of actual nonlinear equations arising from the formfinding problem for prestressed cable nets [5]. The new fractals were found to be beautiful and diverse. Experiments were also performed with some other iterative methods and results were equally fascinating. These new fractals will be referred to as FormFinding Fractals (FFF).

It is difficult to imagine that the formfinding equations represent some special case [6] and it seems highly probable that many other nonlinear equation systems must generate interesting fractals as well.

1.2 Cable net structures

Prestressed cable nets represent rational attractive structures, in particular for long span roofs (Figure 1).

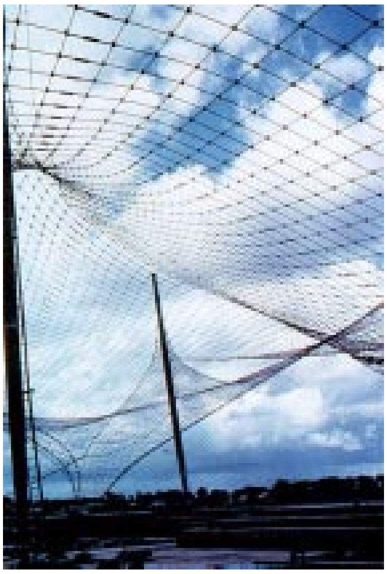


Fig. 1 Example of cable structure [15]

Cables have almost uniform tensile stresses. Forces caused by any external loading are small in comparison with prestressing forces. A disadvantage of cable nets is a requirement of a stiff and strong complementary structures for anchorage of cable ends (similar to frame of a tennis racket), which increases overall cost. Even if the issue of anchorages is taken into account, large span halls with net roofs are still comparatively inexpensive.

In contrast to conventional structures, lack of flexural cable stiffness limits architect's creativity. The shape is determined by equilibrium conditions only. It is however possible to impact on the structural shape indirectly by changing the boundary geometry, by variations of the prestressing force distribution, by modifications of cable arrangements, by loading parts of the structure with some ballast, by adding a flexurally stiff elements etc, ...

The simplest numerical model for approximate shape determination (formfinding) is formulated by neglecting all loadings except prestressing forces. The cable intersection points are called nodes and cable segments between two nodes are referred to as elements. Every node must be connected with three or more elements.

The shape of the cable net in equilibrium minimizes potential energy:

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} P_{i,j} l_{i,j} = \\ = \sum_{i=1}^{n} \sum_{j=1}^{n} P_{i,j} \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2} \qquad (1)$$

where x_i, y_i, z_i are coordinates of the i^{th} node, $P_{i,j}>0$ is the prestressing (tensile) force in the element between nodes i and j, and $l_{i,j}$ is the element length. The sum in Eq. (1) is taken over all pairs of nodes i and j connected by elements. The total number of nodes in the net is n. The free nodes have three translational degrees of freedom (DOF). The anchorage points are fixed nodes. Their DOF are restrained and their coordinates are given in advance as boundary conditions. Clearly the anchorage points must not lie in a single plane - otherwise the entire cable net would lie in the same plane. Some nodes can be partially restrained and may have one or two DOF. As all prestressing forces are tensile, Q is a convex function of coordinates of all free nodes coordinates, and consequently has exactly one real minimum, which can lie in a regular or a singular point of the function. Singularity of the function Q at the minimum point corresponds to the degenerate shape, when at least two nodes coincide and one or more element lengths become equal to zero. In engineering applications, an occurrence of a singular minimum is an indication of a design error.

2. FORMFINDING PROBLEM

In the regular minimum all derivatives of Q with respect to all DOF must be equal to zero. Differentiating function Q with respect to coordinates, a system of equilibrium equations is obtained, three per every free node:

$$\begin{aligned} \frac{\partial Q}{\partial x_i} &= \sum_{j=1}^{n_i} \frac{P_{i,j}(x_i - x_j)}{l_{i,j}} = 0 \\ \frac{\partial Q}{\partial y_i} &= \sum_{j=1}^{n_i} \frac{P_{i,j}(y_i - y_j)}{l_{i,j}} = 0 \qquad i = 1,\ldots,n \qquad (2) \\ \frac{\partial Q}{\partial z_i} &= \sum_{j=1}^{n_i} \frac{P_{i,j}(z_i - z_j)}{l_{i,j}} = 0 \end{aligned}$$

The sums are spread over all pairs (i,j) connected by n_i elements joined at node i. The (real) unknowns are coordinates x_j, y_j and z_j of free nodes and the axial forces $P_{i,j}$. The total number of unknowns is $3n_f+n_e$. The number of free nodes is n_f and n_e is the total number of elements, but the number of equilibrium equations is only $3n_f$. Number of unknowns is therefore greater than a number of equations available, hence the solution is not unique, and n_e additional equations must be given for the problem to be solved. The case with prescribed prestressing forces $P_{i,j}$ [7] is considered here. In this case the additional equations are trivial: $P_{i,j}=const$. If all prescribed prestressing forces are equal, the sum of cable lengths in equilibrium becomes minimal. Such a configuration is called the minimal net - in analogy with minimal surfaces [8, 9].

All formfinding algorithms numerically solve the system of nonlinear algebraic Eqs. (2) by iterative methods, where the Newton-Raphson (NR) method represents a classical approach. A short description of NR method for a single real nonlinear function, and for a system of such functions is given in the Appendix A.

Previous experience and numerical experiments indicate that the convergence of the NR method for the formfinding equations strongly depends on a choice of a starting vector $\boldsymbol{x}_{(0)}$ which must lie inside the "basin of attraction", (BA), usually of an irregular shape. Sometimes, it may be possible that by starting from an apparently quite reasonable approximation of the solution, the iterative process diverges, while on the other hand, the NR algorithm sometimes surprisingly converges starting from a seemingly bad starting approximation. Similar behaviour for some other equations is analyzed in Ref. [10]. However there always exists a small, smooth (non fractal) part of the BA in the vicinity of the solution ("inner basin of attraction"), but its position is unknown in advance.

The hypothesis that the BA of the NR algorithm in the formfinding problem might be a fractal in the $3n_f$-dimensional space is confirmed. For explanation that it is indeed a fractal, an extremely simplified 1D example is given in the Appendix B. In addition it will be shown that some other iterative methods also seem to have fractal BA, however different ones to the BA of the NR method. Methods with larger BA are generally more suitable for the initial stage of the iterative process because the probability to guess a convergent starting vector is greater. After s iteration steps, the s^{th} approximate solution lies safely within the inner BA of the NR method. The rational strategy is therefore to switch then to the NR method or to any other method with a high asymptotic convergence rate.

3. SINGLE NODE CABLE NET EXAMPLES

3.1 The model example formulation of and the Newton-Raphson method

For numerical experiments and visualization, a series of examples treating a cable net with a single free node and three DOF were performed. In some cases, the BA is smooth and uninteresting, but in other examples it is recognized as a fractal. A 3D visualisation of a typical simple BA is shown on Figure 2 where the "3D-pixels" are represented as small cubes. All graphics was realised in Mathematica 4.1 [11].

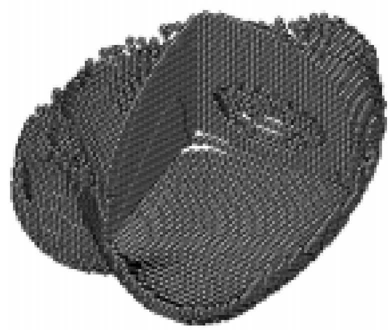


Fig. 2 An example of the 3D Basin of Attraction (BA) for Newton-Raphson (NR) method. A part of the basin is removed.

An example of a cable net with 4 supports and a single free node (Figure 3) will be solved by different iterative methods.

a)

b)

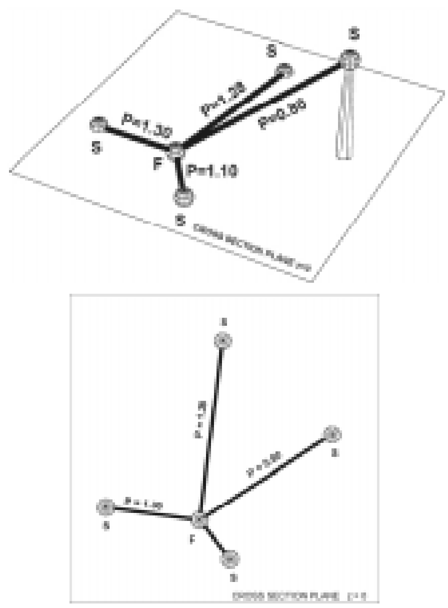


Fig. 3 Starting configuration for a single node "net" used for numerical experiments. Elements are presented as lines and nodes are marked with balls (S - support, F - free node, P - prestressing force): a) perspective; b) plan view.

Absolute units of coordinates and forces are irrelevant, because the ultimate solution depends only on ratio of dimensions and forces. Intersection of BA with the plane $z=0$ for all methods considered is visualized (Figure 4). Coordinates of the supports and prestressing forces are shown in Table 1. The eventual equilibrium position of the free node for given data is: $x=3.5447$, $y=3.5642$, $z=0.3623$, and the iterative procedure is repeated for every pixel on a square 480×480 pixels within $z=0$ plane. Figures 5 to 11 corresponding to different methods have all the same scale for easier visual comparison of the BA shape and size. The convergence criterion is chosen so that the norm of the unbalanced forces is $|\boldsymbol{r}|<10^{-6}$. An iteration is considered as divergent if at least one of the folowing criteria is satisfied:

– Current solution in a given iteration step lies in the neighbourhood of one of the supports - singular point of function Q ($l_i<10^{-6}$).
– Iterated point is too far ($l_i>10^{6}$). All sufficiently distant intermediate solutions produce divergence.
– The Jacobian matrix in a given step is almost singular ($det\ \boldsymbol{J}<10^{-6}$).
– Iteration does not satisfy criterion after N steps ($N=500$ adopted here. It may be that some pixels can be erroneously considered as divergent although they only need larger iterations to converge. In numerical experiments this criterion was not relevant.)

Table 1 Prestressing forces and coordinates of the supports

Support number i	Coordinates x_i	y_i	z_i	Prestressing forces
1	4.4	2.5	0.0	1.10
2	4.2	8.5	0.0	1.28
3	1.0	3.9	0.0	1.30
4	7.2	5.9	3.0	0.80

Different colours stand for different number of iterations needed to satisfy covergence criterion, however colouring scheme is not consistent for all methods considered. For each fractal this scheme is selected separately. As a reference solution the BA for NR algorithm is visualized.

3.2 Modified Newton-Raphson method

The property of quadratic convergence of the standard NR guarantees a small number of iteration steps for achieving prescribed covergence criterion, if starting vector lies within the basin of attraction. Unfortunately every step requires many numerical operations: as the generation of the Jacobian matrix and the solution of the linear(ized) system of equations is required. Computer time and memory needed for calculations can be large. One of usual techniques for reducing the total "cost" is the so-called Modified NR method (MNR). A Jacobian matrix is generated only in the first step, and usually decomposed (inversion is numerically uneconomical for large matrices). In the subsequent steps $\boldsymbol{J}_{(i)}$ is approximated by $\boldsymbol{J}_{(0)}$. Number of numerical operations per iteration is greatly reduced, but the convergence rate is no longer quadratic, therefore the number of iterations increases and the basin of attraction for the model cable net problem is smaller (Figure 5).

The good compromise between the NR and MNR can be a periodic update of the approximate matrix by calculation of the proper Jacobian matrix after every m steps. The MNR is efficient for a weakly nonlinear problems where the starting Jacobian matrix is a good approximation of subsequent ones. On the contary, the formfinding problem is highly nonlinear, and MNR has a small BA and a poor asymptotic convergence rate.

3.3 Underrelaxation

This procedure is a simple extension of the standard NR or MNR algorithm, where every NR or MNR solution increment is multiplied by a factor $\omega<1.0$. For a model cable net problem the underrelaxation method increases the basin of attraction in comparison to the NR or MNR (Figures 6a-6b). Of course, the quadratic convergence property of NR method is lost.

3.4 Line search

An optimal underrelaxation factor ω is different for every iteration step. Line Search (LS) is an improvement to the use of underrelaxation with NR or MNR based on the calculation of ω within every step. The increment multiplied by ω satisfies equilibrium in the direction of the increment vector. The dot product of the increment vector and the residual force vector is therefore equal to zero. The factor ω is found numerically by any robust method for solving a single nonlinear equation, it is chosen here the bisection method. (It should be noted that the NR method for a single equation would not be a good choice because of a possible irregular behaviour, similar to the example from the Appendix B.) The BA for the LS is usually much larger than the one for the standard NR or MNR. An iteration process with LS generally requires less steps, especially in the early stage of iterative process.

If we apply LS to the example from the Appendix B (or for any problem with a single equation with a monotonic function and only one solution) the basin of attraction will be extended to a complete domain, and only one global step will be sufficient (albeit with more inner bisection steps; Figures 7a-7c).

3.5 Steepest descent method with line search

The steepest descent method SD is the simplest and the most "naive"of all gradient methods. The new step is chosen in the direction of the current residual.

For the cable net model problem a combination of SD with LS is very efficient, but it is known that such a procedure is not suitable for realistic problems, as even for linear equations with many unknowns its convergence is very slow. Nevertheless the method is interesting for fractal experiments (Figure 8).

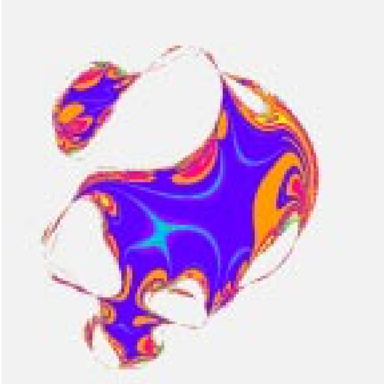


Fig. 4 Simple net problem: BA for the NR method

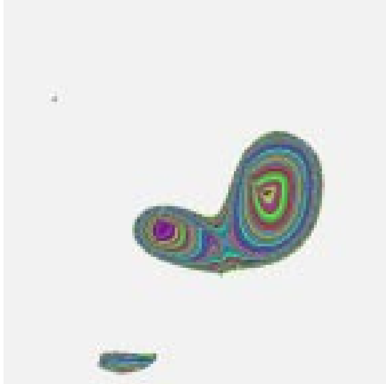
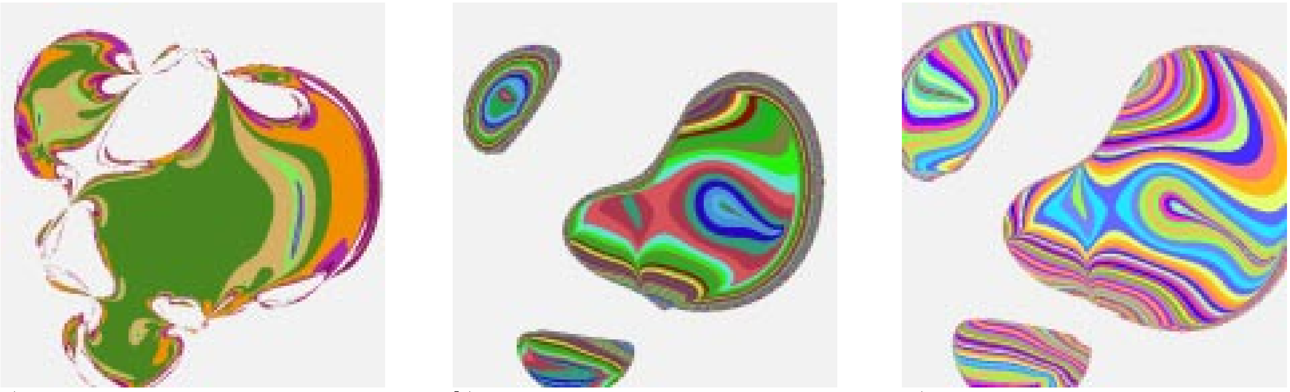
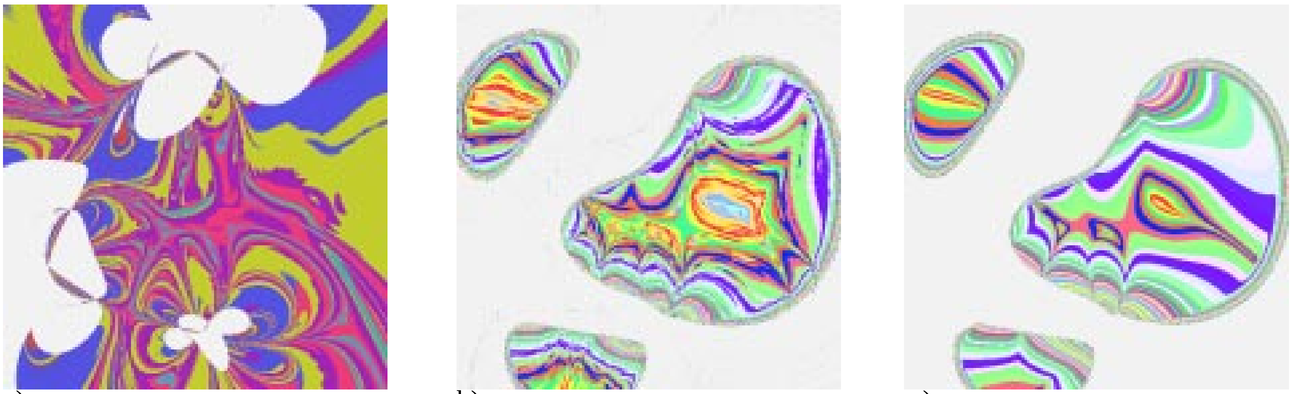


Fig. 5 Simple net problem: BA for Modified Newton-Raphson (MNR) method



a) b) c)

Fig. 6 Simple net problem: Underrelaxation. BA for: a) the NR, ω = 0.6; b) the MNR, ω = 0.6; c) the MNR, ω = 0.4.



a) b) c)

Fig. 7 Line search (LS): a) with the NR; b) with the MNR; c) MNR with the LS in the first five steps

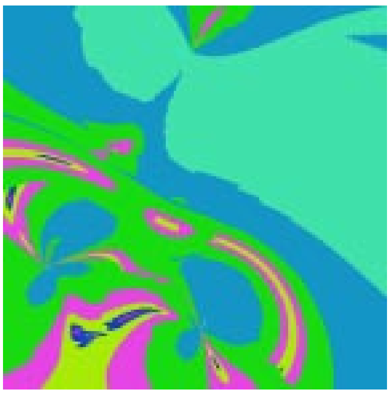


Fig. 8 Simple net problem: Steepest Descent (SD) with the LS

3.6 Line search with a slack criterion

In many applications, a slack criterion for the LS is a good enough approximation. The factor $\beta<1.0$ is introduced and it is sufficient that the new unbalanced force $\boldsymbol{r}_{(i)}$ in the direction of the step is reduced in comparison with the old one $\boldsymbol{r}_{(i-1)}$: $\boldsymbol{r}_{(i)} \le \beta\, \boldsymbol{r}_{(i-1)}$ (BA for the cable model problem is illustrated in Figures 9a-9c).

3.7 Force density method

The force density method will be developed as a problem specific procedure for formfinding problem with prescribed prestressing cable forces [12, 13]. The ratio of the prestressing force to length in Eq. (2) is assumed to be constant in every step: $P_{i,j}/l_{i,j}=D_{i,j}=const.$ This ratio is reffered to as the force density (FD), and Eqs. (2) can be rewritten:

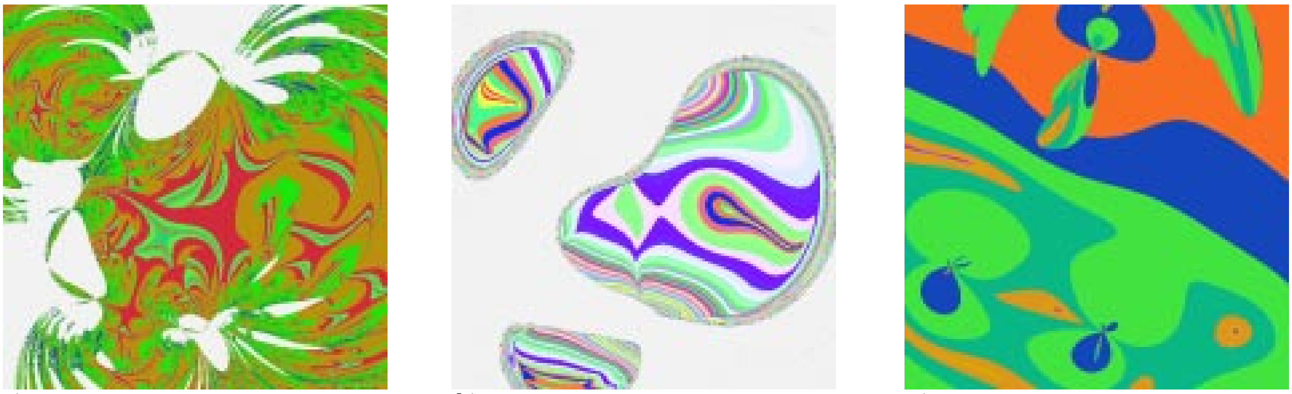
$$\begin{aligned} &\sum_{j=1}^{n_i} D_{i,j}\left(x_i - x_j\right)=0 \\ &\sum_{j=1}^{n_i} D_{i,j}\left(y_i - y_j\right)=0 \\ &\sum_{j=1}^{n_i} D_{i,j}\left(z_i - z_j\right)=0 \end{aligned} \tag{3}$$

leading to three uncoupled systems of linear equations with equal matrices. After solution of all three systems an exact equilibrium solution of the formfinding problem is obtained (neglecting numerical errors) without iteration, but computed forces in elements are not equal to the prescribed ones. In order to achieve the prescribed forces, an iteration is therefore necessary. The new lengths are calculated from the current coordinates:

$$l_{i,j} = \sqrt{\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2 + \left(z_i - z_j\right)^2}$$

and then the new force densities $D_{i,j}$ are computed as ratios of prescribed forces and current lengths. For the model cable net problem, the starting configuration is chosen as geometrically admissible to enable visualisation. (In practical formfinding calculations, the iteration may start from the choice of the element lengths which are not necessarily geometrically admissible. For simplicity, all element lengths can be chosen as unit length.)

The BA for the FD extends to the entire solution space, except for the small neighbourhood in the vicinity of singular points (supports). With its smoothing property, FD method is very suitable for the initial stage of the formfinding algorithm, however it is not optimal for the final stage close to solution convergence. By increasing the number of FD steps the divergent region around singular points is further reduced (Figures 10a-10c).



a) b) c)

Fig. 9 LS with a slack criterion, β=0.5: a) with the NR; b) with the MNR; c) with the SD

a) b) c)

Fig. 10 Starting with the Force Density (FD) method and switch to NR: a) one step of FD; b) three steps of FD; c) ten steps of FD

As a consequence the smoothing property of the FD method less attractive basins of attraction are generated or the fractal nature may even be lost (Figure 11h).

4. SOME ATTRACTIVE FRACTALS

The FFF are interesting, diverse, and attractive, and can clearly be studied independently of the underlying formfinding problem. Attractive fractals can be generated with formfinding algorithms in 2D, omitting the third variable *z* and the corresponding equation, and experimenting with data variations.

Some selected BA with four axes of symmetry are shown in Figures 11a-11i. The 4 or 8 ancorage points are symmetrically arranged. The solutions of the "formfinding problems" are trivial: the equilibrium position of the free node lies in the centre of symmetry.

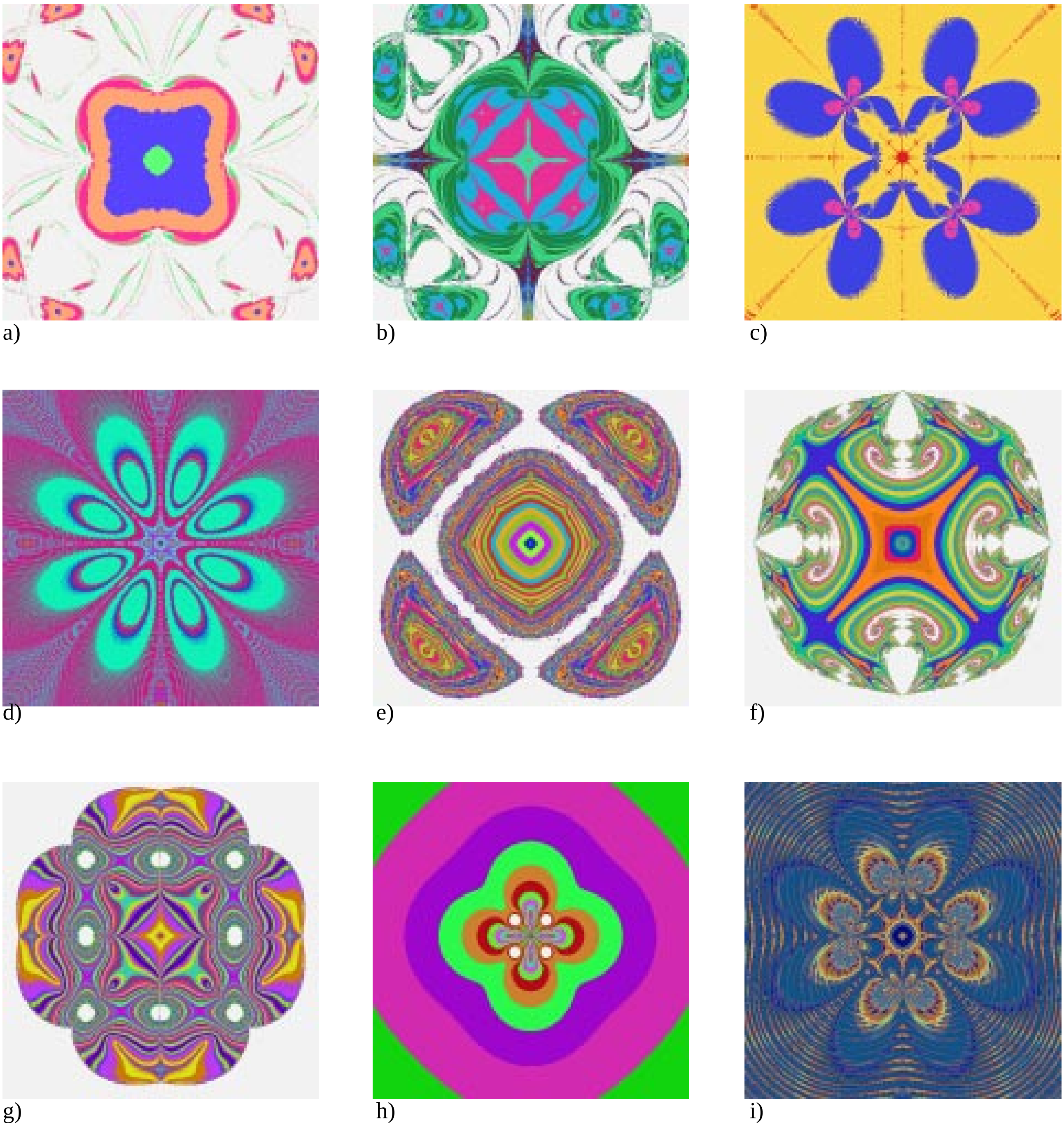


Fig. 11 The BA with four axes of symmetry: a) the NR; b) the NR, ω=0.85; c) the NR and LS, β=0.10; d) SD and LS, β=0.10; e) the MNR and LS, β=0.3, ω=1.10; f) the NR, ω=1.70; g) the MNR, ω=0.45; h) one step of FD and NR, ω=1.40; i) NR and LS, β=0.1

5. CONCLUSION

5.1 Explanation of anomalies in convergence

Although a primary objective to explain the strange convergence of NR and some other methods for small systems was achieved, the fractal nature of the associate BA can only partially explain anomalies in the course of solving large nonlinear systems, because it is coupled with other phenomena associated with large systems e.g. the numerical instability.

5.2 The fractal generation by iterative method for systems of nonlinear equations

The discovery of FFF was an accidental "side effect" following a convergence analysis of a particular system of formfinding nonlinear equations. The fractal (or at least irregular) basins of attraction have been anticipated, but the results of the actual numerical and graphical experiments were fascinating.

Many variations of formfinding algorithms can be interesting for pursuing fractal experiments. For instance, it would be probably interesting to play with the formfinding algorithms in a space with a single free node in more than three dimensions. Of course, the described procedures can be used for other systems of equations, and it can be argued that the systems which are difficult to solve are probably good candidates for fractal generation. Different singularities (or near singularities) of individual functions, the local (near) singularities of Jacobian matrix, "meandering" functions with many inflections and other irregularities usually create difficulties when equations need to be solved, however this is associated with the expected generation of interesting fractals.

5.3 De-mystification of the exclusive link between fractals and complex numbers

It has been shown that assumption alluded to earlier that the complex arithmetic is necessary for the generation of analytic fractals is false! Hence, just as complex numbers may have "magic power", the examples in this paper show, that the real numbers are "magic" too! The analytic fractals can be generated without complex arithmetic.

6. APPENDIX

A Newton-Raphson method for the single real equation and generalisation on systems

For 1D case - individual nonlinear equation $f(x)=0$ the procedure starts from a chosen value $x(0)$, and i^{th} iterative step is:

$$x_{(i)} = x_{(i-1)} - \omega \frac{f\left(x_{(i-1)}\right)}{f'\left(x_{(i-1)}\right)} \tag{4}$$

where, ω is the underrelaxation (overrelaxation) factor $0.0<\omega<2.0$. If $\omega<1.0$ it is usually called "underrelaxation", and if $\omega>1.0$ "overrelaxation".

For the standard NR method $\omega=1.0$ and it is omitted from Eq. (4). $f'(x)$ means derivative of $f(x)$ with respect to x. The subscript in parentheses denotes the ordinal number of the iteration step. The procedure stops when some numerical criterion is satisfied, for example $\delta_n=\left|f(x_{(n)})\right|<\varepsilon$, where ε represents an acceptable error in numerical solution. The value $x_{(n)}$ is then accepted as an approximate solution to $x\approx x_{(n)}$. If the criterion is not satisfied after n steps (n is a chosen, large enough integer constant), it can be estimated (with reasonable probability) that the numerical process diverges. By repeating the iteration process with different starting values $x_{(0)}$, the process can converge to different solutions, or it can diverge.

An advantage of the standard NR method (with $\omega=1.0$) is the quadratic convergence. It means that in the vicinity of solution, the error in the i^{th} iteration step is proportional to the square of error in the previous $(i\text{-}1)^{th}$ step: $\left|\delta_{(i)}\right|=\gamma\left|\delta_{(i-1)}\right|$ (γ is a positive constant).

The NR method could be generalized for systems of n nonlinear equations:

$$\begin{gathered} f_1(x_1,x_2,\dots x_r,\dots x_n)=0, \\ \vdots \\ f_r(x_1,x_2,\dots x_r,\dots x_n)=0, \\ \vdots \\ f_n(x_1,x_2,\dots x_r,\dots x_n)=0 \end{gathered} \tag{5}$$

Scalar values $x_{(0)}$, $x_{(i)}$, $x_{(i-1)}$, $f(x)_{(i-1)}$ from one-dimensional case, here are replaced by the vector-columns $\boldsymbol{x}_{(0)}$, $\boldsymbol{x}_{(i)}$, $\boldsymbol{x}_{(i-1)}$ and $f(\boldsymbol{x}_{(i-1)})$ and $f'(\boldsymbol{x}_{(i-1)})$ by the Jacobian matrix $\boldsymbol{J}_{(i-1)}$. Elements of the Jacobian matrix are $J_{r,s(i)}=\partial f_{r(i)}/\partial x_s$. The NR iteration is:

$$\boldsymbol{x}_{(i)}=\boldsymbol{x}_{(i-1)}-\omega\,\boldsymbol{J}^{-1}_{(i-1)}\ \boldsymbol{f}(\boldsymbol{x}_{(i-1)}) \tag{6}$$

Actual calculation of the inverse of Jacobian matrix $\boldsymbol{J}^{-1}_{(i-1)}$ is not necessary, but it is sufficient to find the product $\boldsymbol{J}^{-1}_{(i-1)}\ \boldsymbol{f}(x_{(i-1)})$ by solving a system of linear(ized) equations. The stopping criterion is: $|\boldsymbol{f}(x_{(i-1)})|<\varepsilon$, where $|\boldsymbol{f}(x_{(i)})|$ denotes a vector norm, for example the Euclidean norm:

$$\left|\boldsymbol{f}\left(\boldsymbol{x}_{(i)}\right)\right| = \sqrt{f_1\left(x_{(i)}\right)^2 + f_2\left(x_{(i)}\right) + \dots +^2\ f_r\left(x_{(i)}\right)^2 + \dots + f_n\left(x_{(i)}\right)^2} \tag{7}$$

The property of quadratic convergence was also proved for systems of equations [14].

B Newton Raphson method on a single DOF formfinding equation

A formfinding equation with one unknown is:

$$f(x)=\sum_{j=1}^{n_i}\frac{P_j\left(x-x_j\right)}{\sqrt{\left(x-x_j\right)^2+\left(y-y_j\right)^2+\left(z-z_j\right)^2}}=0 \quad (8)$$

where x, y, z (without subscripts) are the unknown coordinates of the single free node, and x_j, y_j, z_j are the coordinates of the j^{th} support. If coordinates y and z are fixed (the only degree of freedom is translation in the x coordinate direction), subexpression $(y-y_j)^2+(z-z_j)^2$ is a positive constant c_j. The Eq. (8) is then rewritten as:

$$f(x)=\sum_{j=1}^{n_i}P_j\frac{x-x_j}{\sqrt{\left(x-x_j\right)^2+c_j}}=0 \quad (9)$$

As an illustration an antisymmetric function is generated by the following set of data: $n_i=3$ (number of elements), $P_1=P_2=P_3=1.0$ (prestressing forces), $x_1=5.0$; $x_2=0.0$; $x_3=5.0$; $(y-y_1)^2+(z-z_1)^2=c_1=0.5$; $(y-y_2)^2+(z-z_2)^2=c_2=1.0$; $(y-y_3)^2+(z-z_3)^2=c_3=0.5$. After substitution in Eq. (9):

$$f(x)=\frac{x-5.0}{\sqrt{(x-5.0)^2+0.5}}+\frac{x}{\sqrt{x^2+1.0}}+ \\ +\frac{x+5.0}{\sqrt{(x+5.0)^2+0.5}}=0 \quad (10)$$

This equation has an obvious real solution: $x=0$.

The single iteration step can be interpreted as the "NR function":

$$NR(x)=x-\frac{f(x)}{f'(x)} \quad (11)$$

Just as the i^{th} step $x_i=NR(x_{(i-1)})$ maps the point $x_{(i-1)}$ into $x_{(i)}$, the same can be appied to intervals. If X is an interval, $NR(X)$ is the set of maps of all points within X (or the map of set). The ratio of the map of a small interval $NR(\Delta)$, $(\Delta\rightarrow 0)$ in comparison with the original length Δ is:

$$\lambda=\frac{\partial NR(x)}{\partial x}=f(x)\frac{f''(x)}{f'(x)^2} \quad (12)$$

where x is a point within the interval Δ. Although an inverse function $NR^{-1}(x)$ is not unique the inverse mapping is defined: $NR^{-1}(x)$ is a set of points y satisfying equation $NR(y)=x$. The geometric interpretation is that it is a set of abscisas of points on the curve $f(x)$ with tangents passing through the point x.

Consequently the set of all points on the x axis can be divided in 3 subsets:

C is the subset of converging points: Starting from any point inside subset C, the NR iteration converges to the solution $x=0$. This subset is 1D fractal.

D is an analogous subset of diverging points tending to $\pm\infty$. The divergent subset is also 1D fractal.

L is the set of cycling points. Starting from a point in the subset L, iteration can have an initial aperiodic phase, and then reach a periodic cycle (Figure 13). Number of different cycles is infinite, but they are unstable and it is practically impossible to find the cycle by the NR iteration.

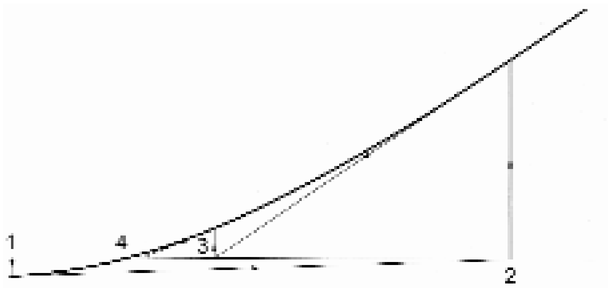


Fig. 12 Visualisation of the three steps of NR method for the single equation

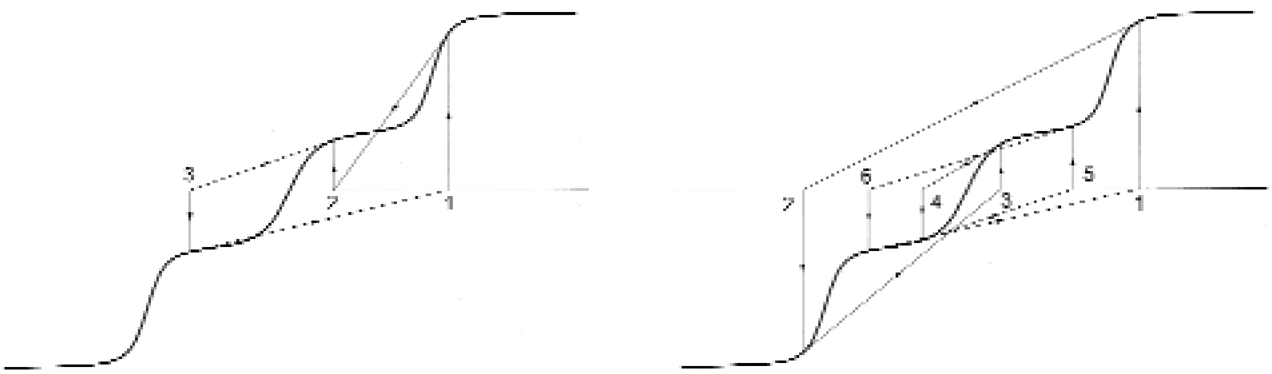


Fig. 13 Two examples of cycles: a) three steps; b) six steps

It is evident that points $NR^k(x)$ for a positive or negative integer k must belong to the same subset as the point x. The m-step cycles can be found by solving equations $NR^m(x)=x$, where $NR^m(x)=x$ means $NR(NR(...(x)))$ m times.

For example, to determine the values of the two step cycles the equation $NR^2(x)=x$ must be solved (for the antisymmetric illustrative example it is easier to solve $NR(x)=-x$).

The important antisymmetric two step cycles are: $A_1, A_2, A_1,...$, and $B_1, B_2, B_1,...$, where $A_1=-A_2=1{,}0129$ and $B_1=-B_2=6{,}0213$ (Figure 14a).

The open interval inside the first of these cycles $X_{c1}=\langle A_2<x<A_1\rangle$ is the classical (continuous) part of the basin of attraction ("inner basin of attraction"), coloured green on the Figure 14a. The points from this basin are convergent.

In analogy the positive and negative part outside the cycles $X_{d1}=\langle -\infty<x<B_2\rangle \wedge \langle B_1<x<\infty\rangle$ are classical (continuous) parts of basin of repulsion, here coloured red. With an initial value from this set the NR iteration oscillates between left and right part with the increasing amplitudes. Every point from remaining intervals $X_u\langle B_1<x<A_1\rangle$ and $X_u'\langle A_2<x<B_2\rangle$ belong to one of sets C, D or L.

In the Figure 14b the points of the convergent subset X_{c2} (second generation) are also coloured green. This is a subset of convergent points from X_u and X_u' which satisfy the relation $NR(X_{c2})=X_{c1}$, i.e. the set of points which after one NR iteration step are mapped in X_{c1}. It is interesting to observe that the NR iteration starting from a special point from each interval converges to the exact solution in one step.

The analogous, divergent subset X_{d2} is coloured red. This is a subset of divergent points from X_u and X_u' which satisfy relation $NR(X_{d2})=X_{d1}$, i.e. the set of points which after one NR step are mapped in X_{d1}.

In the Figure 14c the points of the convergent subset X_{c3} are again coloured green. This is a subset of points from X_u and X_u' which satisfy relation $NR(X_{c3})=X_{c2}$, i.e. the set of points which after one NR step are mapped into X_{c2}. The analogous, divergent subset X_{d3} is coloured red. This is a subset of convergent points from X_u and X_u' which satisfy relation $NR(X_{d3})=X_{d2}$, i.e. the set of points which after one NR step are mapped in X_{d2}. Parts of x axis, not coloured in Figure 14c belong to the third or higher generation. This series of convergent and divergent subsets is infinite.

The self-similarity can be easily deduced:

An interval P_1 is chosen and its map $P_2=NR(P_1)$. If the mapping for the whole interval is not continuous or unique, a subinterval can be chosen with continuous mapping. On the interval P_2 a pattern of convergent-divergent cyclic points is noticed. This pattern must be also (inversely) mapped onto P_1 - it is similar but distorted. The analogous deduction is also possible in more dimensions.

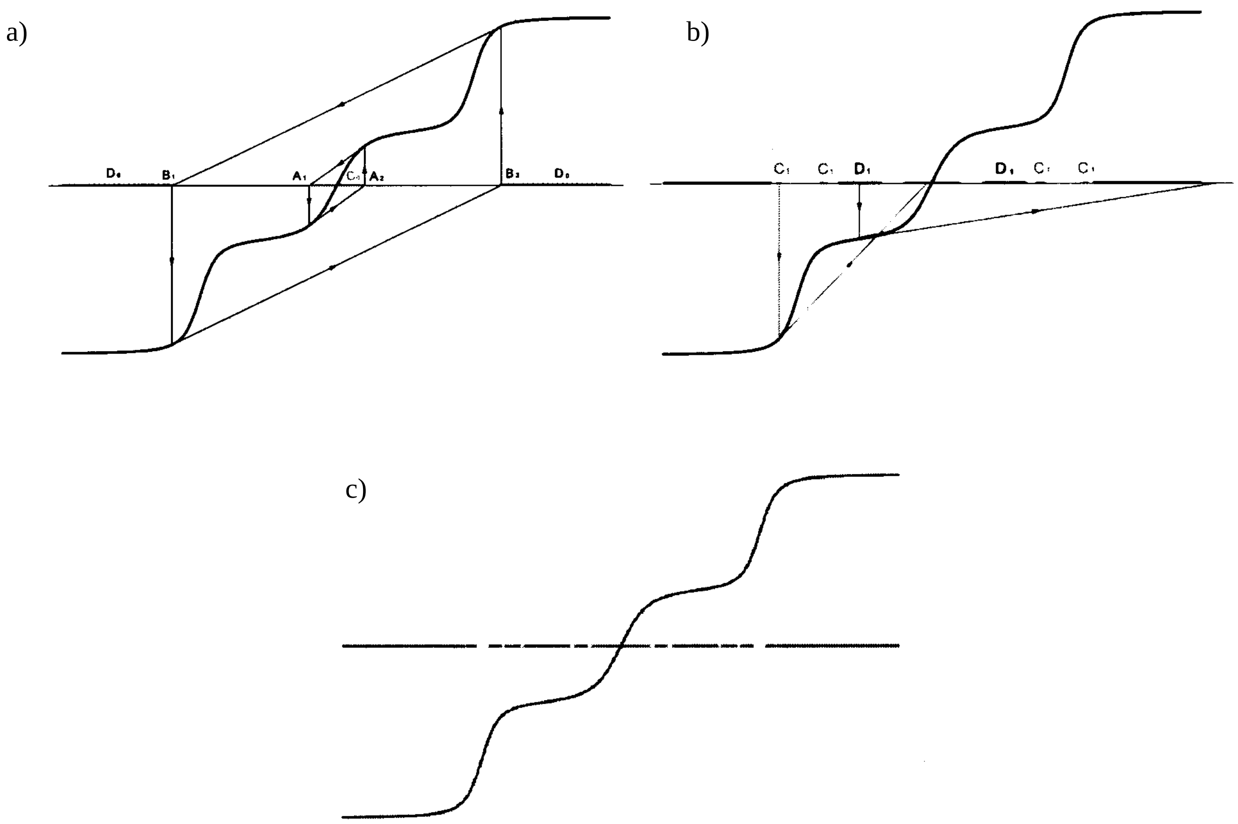


Fig. 14 Basins of attraction and repulsion: a) classic solution; b) and c) generation of additional basins: b) first generation; c) first and second generation (symbols are omitted)

7. REFERENCES

[1] B.B. Mandelbrot, *The Fractal Geometry of Nature*, W.H. Freeman, New York, 1977.

[2] M. Barnsley, *Fractals Everywhere*, Academic Press, San Diego, 1988.

[3] G. Genta, Experiments on image generation using the Newton-Raphson algorithm, Department of Mechanics, Politecnico di Torino, Torino, http://www.giancarlogenta.it/fractal.pdf

[4] G. Zito, Drawing by accident, http://www.ba.infn.it/~zito/ufattr/ufattr. html

[5] J. Dvornik and D. Lazarević, Pretensioned flexible structures made of cables and fabric, *Građevinar*, Vol. 47, No. 4, pp. 185-199, 1995. (in Croatian)

[6] B. Tabarrok and Z. Qin, Nonlinear analysis of tension structures, *Computers & Structures*, Vol. 45, No. 5/6, pp. 973-984, 1992.

[7] J. Dvornik and D. Lazarević, Flexible structures made of fabric and cables, *Građevni godišnjak*, pp. 239-297, 1997. (in Croatian)

[8] R. Courant, *Dirichlet's Principle, Conformal Mapping and Minimal Surfaces*, Interscience, New York, 1950.

[9] J.C.C. Nitsche, Plateau's problems and their modern ramifications, *The American Mathematical Monthly*, Vol. 81, No. 9, pp. 945-968, 1974.

[10] R.H. Byrd, M. Marazzi and J. Nocedal, On the convergence of Newton iteration to non-stationary points, Report OTC 2001/01, Optimization Technology Center, Northwestern University, Evanston, IL 60208, USA, 2001.

[11] S. Wolfram, *Mathematica, A system for Doing Mathematics by Computer*, Addison-Wesley Publishing Company, Massachusetts, 2000.

[12] H.- J. Sheck, The force density method for formfinding and computation of general networks, *Computer Methods in Applied Mechanics and Engineering*, Vol. 3, pp. 115-134, 1974.

[13] M.R. Barnes, Computer aided design of the shade membrane roofs for EXPO 88, *Structural Engineering Review*, Vol. 1, pp. 3-13, 1988.

[14] E. Isaacson and H.B. Keller, *Analysis of Numerical Methods*, Dover Publications, Inc., New York, 1994.

[15] F. Otto, B. Rasch, *Finding Form*, Edition Axel Menges, 1995.

FRAKTALI I 'FORMFINDING' - ČAROLIJA S REALNIM BROJEVIMA

SAŽETAK

Postupak određivanja ravnotežnog oblika mrežaste konstrukcije od prednapregnute užadi (engl. formfinding), može se provesti pomoću klasične Newton-Raphsonove iterativne metode. Konvergencija strogo ovisi o izboru početne aproksimacije, koja mora ležati unutar unaprijed nepoznatog područja konvergencije. Iz iskustva je poznato da je to područje nepravilno i diskontinuirano pa se naslutilo da bi moglo imati svojstva fraktala. Da bi se hipoteza potvrdila i točnije razjasnila, proveden je niz numeričkih pokusa na vrlo pojednostavljenim primjerima mreža. Nacrtani presjeci dobivenih trodimenzionalnih fraktala su vrlo zanimljivi i raznoliki. Time se pokazalo da fraktali mogu nastati pri operacijama samo s realnim brojevima. Nakon toga provedeni su i pokusi uz pomoć drugih iterativnih algoritama, također s vrlo zanimljivim rezultatima, kako sa stanovišta rješavanja sustava nelinearnih jednadžbi, tako i sa stanovišta fraktalne geometrije. Očekuje se da bi pokusi s iterativnim metodama iz ovog rada (osim metode "gustoće sila" specifične za formfinding problem) i slični pokusi s drugim metodama mogli pomoći razjašnjavanju ponašanja mnogih sustava nelinearnih jednadžbi.

Ključne riječi*: konstrukcije od mreže užadi, određivanje oblika, područje konvergencije, fraktali, iterativne metode, nelinearne jednadžbe.*