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Non-linear analysis of time-dependent response of concrete structures

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SUMMARY

This paper deals with mechanical and rheological properties of concrete and with numerical modelling of the constitutive law of concrete taking into consideration its ageing, decomposition and non-linear creep. A developed method of general non-linear analysis of the time-dependent response of concrete structures is also presented. It is limited to reinforced, prestressed or composite plane frames. In addition to the influences of rheology, including the non-linear creep of concrete, the time-variable structural system can also be taken into account. Therefore, the presented method of analysis is suitable for predicting the time development of displacements and internal forces, and for making an estimation of the safety of concrete structures in any intermediate or final constructional phase. The applicability of the prepared software is briefly presented by example.

***Key words**: concrete, rheology, ageing, shrinkage, creep, non-linear analysis, time-dependent response, numerical modelling, changeable structural system.*

1. INTRODUCTION

In addition to material and geometrical non-linearities, usually included in a computational non-linear analysis of structures, the gradual construction is another important contribution to their non-linear behaviour [1, 2]. In reinforced or prestressed concrete structures, an additional treatment is necessary due to the rheology of material.

For the computational consideration of the influence of sustainable changeable high stress levels on the compressive strength of concrete, this paper gives a relatively simple model, where the whole history of previous loadings is taken into account with the parameter of the already exploited limit time. With the help of this parameter it is possible to determine the compressive strength of concrete in an arbitrary time. As a basis for the development of the model and for the evaluation of the introduced model constant, the provisions of the CEB-FIP Model Code 1990 [3] were used, appertaining to the decreasing compressive strength of concrete at constant high stress.

The finite element method (FEM) was applied to the presented method of analysis of time-dependent response of concrete structures considering together the rheology, variable structural system, cracks and geometrical non-linearity. In modelling the viscoelastic properties of ageing concrete, the generalised Kelvin model was used which, in addition to experimentally acquired functions of concrete creep, can also approximate the creep functions of concrete according to the CEB-FIP Model Code 1990 [3]. The non-linearity of creep was taken into account through functions which, depending on the momentary stress level, increase the speed of creep [4-6]. For the time integration, the method with discretization of inelastic strain, considering the linear interpolation of stress within the time interval, as presented by Bažant [7], was partly modified. The NONFRAN computer program [8] for general static analysis of planar frame structures considering material and geometrical non-linearity was used as a basis for software development. The elaborated software allows the consideration of gradual changing of a structural system through constructional phases.

2. MECHANICAL AND RHEOLOGICAL PROPERTIES OF CONCRETE

2.1 Ageing and decomposition of concrete

Time dependent increase of concrete compressive strength $f_{cm}(t)$ and modulus of elasticity $E_{cm}(t)$ due to ageing [9] are in CEB-FIP Model Code 1990 [3] defined by the following equations respectively:

$$f_{cm}(t) = \beta_{cc}(t) \cdot f_{cm}\,;\ \beta_{cc}(t) = e^{\left\{s\left[1-\sqrt{\frac{28}{t}}\right]\right\}} \tag{1}$$

$$E_{cm}(t) = \left[\beta_{cc}(t)\right]^{0,3} E_{cm} \tag{2}$$

where f_{cm} is the compressive strength of concrete at the age of 28 days and the influence of the type of cement on the time development of mean concrete compressive strength $f_{cm}(t)$ (Eq. (2)) is taken into account by an additional parameter s. If the temperature differs from $20°C$ the concrete age t in the Eqs. (1) and (2) has to be replaced with the temperature adjusted concrete age t_T according to the Eq. (3), where $T(\Delta t_i)$ is the mean temperature in the time interval Δt_i:

$$t_T = \sum_{i=1}^{n} e^{-\left(4000/\left[273+T(\Delta t_i)\right]-13.65\right)} \cdot \Delta t_i \tag{3}$$

Due to the decomposition of concrete under the influence of high stress, the long-term strength of concrete is approximately *15%* lower than its short-term strength. With the help of a systematic analysis of the results for specimens which failed under long-term loading [10], Graser and Krämer [11] prepared diagrams by assigning to the duration of loading $(t–t_0)$ the appertaining strength, normalized to the concrete strength at the age of 28 days. Individual curves belong to different ages of concrete at the time of loading. If the points in which individual curves reach minimal values are connected, the curve of critical times $(t–t_0)_{crit}$ is achieved.

2.2 Concrete shrinkage

The total shrinkage of concrete consists of:

* Autogenous shrinkage of concrete which is the consequence of chemical binding of water in the concrete in the process of cement hydration;
* Concrete shrinkage due to drying;
* Concrete shrinkage due to the carbonisation of the cement stone.

At the concrete of normal strength ($f_{ck} \le 50\ MPa$), the main part of the shrinkage is caused by concrete drying, while the autogenous shrinkage in the first two days is comparatively small. In high strength concrete ($f_{ck} > 50\ MPa$), the autogenous shrinkage represents, due to low water-cement ratio in the initial stage of cement binding, an important part of the total concrete shrinkage and cannot be neglected [12]. For this reason the shrinkage of high-strength concrete has to be measured from the very beginning of the binding process in the cement so that autogenous shrinkage in the first two days after concreting is also considered [13, 14]. In the presented computational example the shrinkage of normal strength concrete is considered according to the proposal of the CEB-FIP Model Code 1990 [3], in which the concrete shrinkage $\varepsilon_{cs}(t)$ is given in an analytical form as a product of the nominal shrinkage ε_{cs0} and the function β_s, describing the time development of concrete shrinkage with the next equation:

$$\varepsilon_{cs}(t) = \varepsilon_{cs0} \cdot \beta_s(t - t_s) \tag{4}$$

where t is the age of concrete, i.e., the time under consideration, and t_s is the duration of the initial moist curing of concrete.

2.3 Concrete creep

In our method of analysis, the coefficient of linear concrete creep was considered according to the proposal of the CEB-FIP Model Code 1990 [3].

The non-linear creep of concrete, which is inseparably connected with the long-term compressive strength, has to be considered in the case where stresses are larger than *40%* of the short-term concrete compressive strength ($\sigma_c/f_c > 0.4$). In the range of working stresses, the concrete creep is caused mainly by the extraction of the physically absorbed water of the cement stone, while the propagation or formation of microcracks in the contact area or in the cement stone has, at this level of stress, a much lesser influence on the creep. By increasing the stress level, the influence of the propagation and formation of microcracks, at first in the contact area and later on also in the cement stone, increases. Therefore, at stresses between *75* and *85%* of the short-term compressive strength, the share of the basic concrete creep, due to the extraction of the physically absorbed water of the cement stone, in the total creep is already negligible. At this stress level, it can happen that the quantity of released deformation energy is already sufficient for cracks to develop spontaneously. When the stresses achieve this limit, the time-dependent failure of concrete occurs at stresses which are smaller than the short-term strength.

3. MODELLING OF CONCRETE PROPERTIES

3.1 Ageing and decomposition of concrete

The numerical modelling of the influence of long-term changeable high stresses on the compressive strength of concrete [15] is based on the generalisation of the CEB-FIP Model Code 1990 [3] recommendations for concrete structures, which basically apply to constant level of stress. Here, two opposite time effects are included at the same time. At low stress level the concrete strength increases due to favourable influence of ageing, while on the other hand it decreases due to its decomposition resulting from permanent action of high stresses:

$$f_{cm,sus}\left(t,(t-t_0)\right)=\beta_{c,sus}(t-t_0)\cdot f_{cm}(t) \qquad (5)$$

Here, $f_{cm}(t)$ (Eq. (2)) is the short-term concrete strength in time t or at temperature adjusted concrete age t_T (Eq. (1)), and $\beta_{c,sus}(t–t_0)$ is the function of the duration of high stress which takes into account the decrease in concrete strength due to decomposition, defined by next expression:

$$\beta_{c,sus}\left(t-t_0\right)=0.96-0.12\cdot\sqrt[4]{ln\left[72\left(t-t_0\right)\right]} \qquad (6)$$

Using the stress level $\bar{\sigma}^*(t)$ according to the Eq. (7) the validity of Eqs. (5) and (6), which are deemed to be constant stresses according to CEB-FIP Model Code 1990 [3], can also be expanded to the cases of time-dependent stresses:

$$\bar{\sigma}^*(t)=\sigma(t)/f_{cm}(t) \qquad (7)$$

If Δt_u denotes a limited time interval during which the concrete still resists a constant level of stress $\bar{\sigma}^*$, and assuming that at the limit of failure it equals $\beta_{c,sus}$, the length of the limited time interval $\Delta t_u = t–t_0$ can be determined by means of Eq. (6) as:

$$\Delta t_u(t)=\frac{1}{72}e^{\left(8-\frac{\bar{\sigma}^*(t)}{0.12}\right)^4} \qquad (8)$$

Now, the parameter $k(t)$ is introduced, representing the share of the already exploited limited time interval up to the considered time t. Taking into consideration Eq. (8), it is expressed by the following differential equation:

$$dk=\frac{dt}{\Delta t_u(t)}=72\cdot e^{-\left(8-\frac{\bar{\sigma}^*(t)}{0.12}\right)^4}dt \qquad (9)$$

If the previous equation is integrated and the initial condition $k(t = 0) = 0$ is taken into account, an equation for parameter $k(t)$ in the integral form is obtained:

$$k(t)=72\int_0^t e^{-\left(8-\frac{\bar{\sigma}^*(t)}{0.12}\right)^4}d\tau\,. \qquad (10)$$

The exploited limited time interval parameter $k(t)$ thus provides a kind of weight for the dimensionless evaluation of the duration of loading at a certain level (dashed line in Figure 1). The parameter $k(t)$ defines only the shortening, i.e., reduction of the available limited time interval $\Delta t_u(t)$ due to previous loading at high stress levels according to the next equation:

$$\Delta t_{u,red}(t)=\left(1-k(t)\right)\cdot\Delta t_u(t) \qquad (11)$$

and not the reduction of the available strength due to the loading history.

In order to include also the influence of previous high levels of stress on the reduction of concrete strength, a normalized long-term concrete strength $\beta^*_{c,sus}$ is introduced, representing the critical level of loading, where despite infinite duration of loading no concrete strength reduction appears. When assuming, in Eq. (6), *100* years as finite time ($t - t_0 = 100$ years), the normalized long-term strength of $\beta^*_{c,sus}=\beta_{c,sus}(100\,years)=0.7247$ is obtained.

To determine the ultimate stress level, i.e. normalized strength $\beta_{cu}(t)$, the constant k_h is further introduced to include the influence of previous loading at high stress level on the reduction of concrete strength. In the case where the parameter $k(t)$ exceeds the value of *1*, the load bearing capacity of concrete is exhausted, and $\beta_{cu}(t)=0$ applies. For the values of parameter $k(t)$ between *0* and *1*, the ultimate stress level $\beta_{cu}(t)$ is determined on the basis of the condition that the limited time interval of concrete $\Delta t_u(\beta_{cu}(t))$ at a constant stress level $\beta_{cu}(t)$ is equal to the corrected ultimate time of previously loaded concrete (Figure 1). For the concrete previously loaded with changeable high stress levels, the already exploited share of limited time interval comprised in parameter $k(t)$ is taken into account, as well as the linear correction by the constant k_h for stress levels higher than $\beta^*_{c,sus}$:

$$\Delta t_u(\beta_{cu}(t))=$$
$$=\Delta t_u(\beta_{cu}(t))\cdot\left\{k(t)+k_h\cdot\left[\beta_{cu}(t)-\beta^*_{c,sus}\right]\right\}. \qquad (12)$$

It follows from the above equation that the expression in curly brackets should equal *1*, which allows us to define the adequate normalized strength $\beta_{cu}(t)$:

$$\beta_{cu}(t)=\frac{1-k(t)}{k_h}+\beta^*_{c,sus}\ \text{for}\ 0\le k(t)\le 1\,. \qquad (13)$$

The value of the constant k_h, taken into account in the presented analysis of the time-dependent response of concrete structures, is the one defined on the basis of the condition that the normalized compressive strength $\beta_{cu}(t)$ equals *1* when the share of the exploited limited time interval $k(t)$ equals *0*. Taking the value of *0.7247* for $\beta^*_{c,sus}$, the following value of constant k_h is obtained:

$$k = 0 : \beta_{cu}(t) = 1 \rightarrow k_h = \frac{1}{1 - \beta^*_{c,sus}} = 3.63247 \;. \quad (14)$$

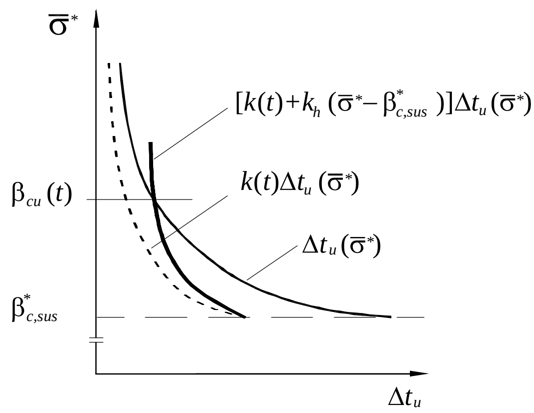

Fig. 1 Graphic representation of the significance of parameter k(t) and of defining normalized concrete strength $\beta_{cu}(t)$

With the history of previous loading at high stress levels taken into account, the concrete strength $f_{cm\beta}$(t) for an arbitrary time t is defined as the product of normalized strength $\beta_{cu}(t)$ (Eq. (13)) and time-dependent concrete strength by taking into account the influence of ageing $f_{cm}(t)$ (Eq. (2)):

$$f_{cm\beta}(t) = \beta_{cu}(t) \cdot f_{cm}(t) \,. \quad (15)$$

3.2 Constitutive law of concrete

The total strain in an arbitrary point of concrete cross-section $\varepsilon(\sigma, t)$ in time t consists of the mechanical strain $\varepsilon_m(\sigma, t)$, resulting from non-linear elastic concrete behaviour under short-term loading, concrete strain due to concrete creep $\varepsilon_c(s, t)$ and concrete strain due to concrete shrinkage $\varepsilon_s(t)$:

$$\varepsilon(\sigma,t) = \varepsilon_m(\sigma,t) + \varepsilon_c(\sigma,t) + \varepsilon_s(t) \quad (16)$$

Modelling of the basic mechanical non-linearity and concrete creep is carried out using the generalised Kelvin model of material (Figure 2), consisting of a non-linear time-dependent spring and the chain of Kelvin bodies ($\mu = 1, 2, ...n$).

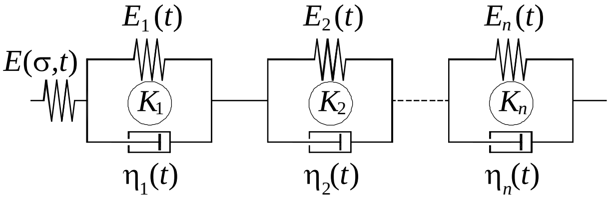

Fig. 2 Generalised Kelvin model

To include all types of non-linearity, the constitutive equations of concrete are written in incremental form. The analysed time period is divided into time intervals, the limits of which are discrete times t_i ($i = 1, 2, \ldots, m$). The analysis of the stress-strain state of the structure is carried out at these discrete times. In deriving the constitutive equations it is assumed that the time distribution of stresses within individual time interval is linear:

$$\sigma(t) = \sigma(t_{i-1}) + \frac{\Delta\sigma_i}{\Delta t_i} \cdot (t - t_{i-1}) \text{ for } t_{i-1} \le t < t_i \,, \quad (17)$$

where $\Delta\sigma_i = \sigma(t_i) - \sigma(t_{i-1}) = \sigma_i - \sigma_{i-1}$ is the change of stress within time interval i, and $\Delta t_i = t_i - t_{i-1}$ is the length of the time interval i considered.

It is assumed that the stress state of an arbitrary point in concrete cross-section at the initial time t_{i-1} is defined by stress σ_{i-1}, and the total strain ε_{i-1}. With the change of stress by $\Delta\sigma_i$ in the time interval Δt_i, the increment of the total constitutive concrete strain $\Delta\varepsilon_i$ can be expressed by the following equation:

$$\Delta\varepsilon_i = \Delta\varepsilon_i(\sigma_{i-1}, \varepsilon_{i-1}, \Delta t_i, \Delta\sigma_i) \,. \quad (18)$$

The increment of constitutive strain of concrete $\Delta\varepsilon_i$ is obtained by summing up the increment of the mechanical part of strain $\Delta\varepsilon_{mi}$, originating from non-linear elastic behaviour of concrete under short-term loading, the strain increment due to concrete creep $\Delta\varepsilon_{ci}$ and the strain increment due to concrete shrinkage $\Delta\varepsilon_{si}$ as:

$$\Delta\varepsilon_i(\Delta\sigma_i, \Delta t_i) = $$
$$= \Delta\varepsilon_{m_i}(\Delta\sigma_i, \Delta t_i) + \Delta\varepsilon_{c_i}(\Delta\sigma_i, \Delta t_i) + \Delta\varepsilon_{s_i}(\Delta t_i) \,. \quad (19)$$

In defining the constitutive law of concrete, two typical states should be distinguished [4, 15]:

* ***In case of relatively low stress levels***, which do not cause decomposition of concrete, the concrete strength $f_{cm\beta}$ and its modulus of elasticity $E_{c\beta}$ increase in time ($df_{cm\beta}/dt \ge 0$, $dE_{c\beta}/dt \ge 0$) due to concrete ageing. For this reason, the modelling of the basic mechanical non-linearity and concrete creep, carried out using the generalised Kelvin model of material (Figure 2), takes into account the mechanical and rheological models according to the theory of concrete solidification.

* ***In case of high stress levels*** the values of mechanical characteristics of concrete can decrease over time, despite the favourable influence of ageing ($df_{cm\beta}/dt < 0$, $dE_{c\beta}/dt < 0$). In this state, the classical mechanical and rheological models are taken into account in modelling the mechanical and rheological behaviour of concrete.

3.2.1 Constitutive law of concrete without considering decomposition

The mechanical and rheological models of material according to the theory of concrete solidification [4, 15] are considered in the generalised Kelvin model of material (Figure 2). The equation $\dot{\sigma} = E(\sigma,t) \cdot \dot{\varepsilon}_m$ is used for the spring which describes the non-linear initial concrete strain, where $E(\sigma, t)$ is the tangent modulus of elasticity at stress σ in time t. By rearranging and integrating this equation, the constitutive Eq. (20) is obtained, which gives the time-dependent mechanical strain of non-linear spring ε_m or its increment within time interval Δt_i:

$$\varepsilon_m(\sigma,t_i) = \varepsilon_m(t_{i-1}) + \int_{t_{i-1}}^{t_i} \frac{d\sigma(\tau)}{E(\sigma,\tau)}$$
$$\Delta\varepsilon_{m_i}(\Delta\sigma_i,\Delta t_i) = \int_{t_{i-1}}^{t_i} \frac{d\sigma(\tau)}{E(\sigma,\tau)} \tag{20}$$

The constitutive equation $\dot{\sigma}_\mu = E_\mu(t) \cdot \dot{\varepsilon}_\mu$ is used for the spring of the μ-*th* Kelvin body, and the constitutive equation $\sigma_\mu = \eta_\mu(t) \cdot \dot{\varepsilon}_\mu$ for the dashpot of the μ-*th* Kelvin body respectively. In this case, the behaviour of the μ-*th* Kelvin body of chain is described by next differential equation:

$$\frac{\dot{\sigma}_\mu}{\eta_\mu} = \frac{1}{\tau_\mu} \cdot \dot{\varepsilon}_\mu + \ddot{\varepsilon}_\mu \tag{21}$$

where τ_μ is retardation time: $\tau_\mu = \dfrac{\eta_\mu}{E_\mu + \dot{\eta}_\mu}$.

The concrete creep strain $\varepsilon_c(\sigma,t) = \sum_{\mu=1}^{n} \varepsilon_\mu(\sigma,t)$ is defined by the sum of strains of individual Kelvin bodies $\varepsilon_\mu(\sigma, t)$, obtained by double integration of the differential equation (21). To simplify the calculation, it is reasonable to consider the retardation times τ_μ as time independent parameters ($\tau_\mu = \eta_\mu / (E_\mu + \dot{\eta}) = const.$), and it is best to select them in advance. In this case, the whole spectrum of the retardation times of the analysed material should be considered. The parameters of dashpots $\eta_\mu(t)$ are defined on the basis of matching as closely as possible the experimental results of concrete creep. The parameters of dashpots define the spring parameters E_μ also indirectly, through the assumed relation $\tau_\mu = \eta_\mu / (E_\mu + \dot{\eta}) = const.$ By taking into account the linear distribution of stresses given by Eq. (17), constant retardation times τ_μ and constant dashpot characteristics $\eta_\mu(t_{i-1/2})$ in the time interval, resulting in constant elastic modulus of spring of the *i-th* Kelvin body $E_\mu(t_{i-1/2})$ within the time interval, the next equation, Eq. (22), for the increment of the creep strain in the *i-th* time interval is obtained:

$$\Delta\varepsilon_{c_i} = g(\bar{\sigma}_{i-1/2}) \sum_{\mu=1}^{n} \left[\dot{\varepsilon}_\mu(t_{i-1}) \cdot \Delta t_i \cdot \lambda_{\mu_i} + \frac{\Delta\sigma_i}{E_{\mu_{i-1/2}}} \left(1 - \lambda_{\mu_i}\right) \right] \tag{22}$$

$$\dot{\varepsilon}_\mu(t_i) = g(\bar{\sigma}_{i-1/2}) \left(\dot{\varepsilon}_\mu(t_{i-1}) \cdot e^{-\frac{\Delta t_i}{\tau_\mu}} + \frac{\Delta\sigma_i \cdot \lambda_{\mu_i}}{E_{\mu_{i-1/2}} \cdot \tau_\mu} \right)$$
$$\mu = 1,2,........,n \tag{23}$$

$$\lambda_{\mu_i} = \frac{\tau_\mu \left(1 - e^{-\Delta t_i/\tau_\mu}\right)}{\Delta t_i} \tag{24}$$

Indices *i-1/2* describe the values of quantities in the characteristic time $t_{i-1/2}$ (at the middle point) of the time interval, $\dot{\varepsilon}_\mu$ is the hidden variable of the *i-th* Kelvin unit (time derivation of strain), as defined by Eq. (23) for the time t_i. As the function $g(\bar{\sigma})$, representing the influence of non-linear creep in dependence of stress level $\bar{\sigma} = |\sigma| / f_{cm\beta}(t)$, the relationship:

$$g(\bar{\sigma}) = exp\left[1{,}5 \cdot (\bar{\sigma} - 0{,}4)\right] \tag{25}$$

from the CEB-FIP Model Code 1990 [3] can be used, being applicable to the stress level $0.4 \le \bar{\sigma} \le 0.6$, or the suggestions by Bažant and Prassanan [4]:

$$g(\bar{\sigma}) = \frac{1 + \bar{\sigma}^2}{1 - \bar{\sigma}^{10}} \tag{26}$$

or Bažant and Kim [5]:

$$g(\bar{\sigma}) = \frac{1 + 3 \cdot \bar{\sigma}^5}{1 - \bar{\sigma}^{10}} \tag{27}$$

which can also be considered as a rough estimation in the area of higher stress. $f_{cm\beta}(t)$ is the average compressive strength of concrete in time t defined with Eq. (15).

The first element in the brackets of Eq. (22) arises from the influence of the history of previous loading, while the second one represents the influence of stress change $\Delta\sigma_i$ in the treated time interval. The advantage of such formulation of the concrete creep law is mainly caused by the fact that the entire history of previous loading is included in the time derivatives of the strains of Kelvin bodies $\dot{\varepsilon}_\mu$ at the end of the previous time step. Thus, in the case of an equilibrium state of a structure at the time t_i, the time derivations of strains $\dot{\varepsilon}_\mu$ for all Kelvin bodies in the chain, needed for the next time interval, should also be calculated according to Eq. (23).

The increment of a stress-independent concrete strain due to shrinkage in the *i-th* time interval (t_{i-1},t_i), can be defined by Eq. (4), taking into account the adequate times of the analysed interval:

$$\Delta\varepsilon_{s_i}(\Delta t_i) = \varepsilon_{cs}(t_i) - \varepsilon_{cs}(t_{i-1}) = \\ = \varepsilon_{cs0} \cdot \left[\beta_s(t_i - t_s) - \beta_s(t_{i-1} - t_s)\right]. \tag{28}$$

3.2.2 Constitutive law of concrete by considering decomposition

Long lasting high stress levels cause the deterioration of mechanical characteristics of concrete despite the favourable influence of ageing. In this case for the modelling of the mechanical and rheological behaviour of elements of the generalised Kelvin model of material (Figure 2) [4, 15] the classical mechanical and rheological models of material are used. The non-linear spring, describing the initial non-linear concrete strain, is considered as a hyperelastic material, where strain depends on the current stress σ and time t. By taking into account the constitutive equations $\varepsilon_m = \varepsilon_m(\sigma, t)$ and by considering linear distribution of stresses within a time interval (Eq. (17)), the next equation for the increment of mechanical strain is obtained:

$$\Delta\varepsilon_{m_i} = \left[\varepsilon_m(\sigma_{i-1} + \Delta\sigma_i, t_i) - \varepsilon_m(\sigma_{i-1}, t_{i-1})\right] \tag{29}$$

For the spring and the dashpot of the μ-*th* Kelvin body the constitutive equations $\sigma_\mu = E_\mu(t) \cdot \varepsilon_\mu$ and $\sigma_\mu = \eta_\mu(t) \cdot \dot{\varepsilon}_\mu$ respectively are used. The behaviour of the μ-*th* Kelvin body is then described by a differential equation $E_\mu \cdot \varepsilon_\mu + \eta_\mu \cdot \dot{\varepsilon}_\mu = \sigma$. The integration of this equation yields the strains of individual Kelvin bodies $\varepsilon_\mu(\sigma, t)$, and the concrete creep strain ε_c is obtained by summing up the individual contributions of all Kelvin bodies:

$$\varepsilon_c(\sigma, t) = \sum_{\mu=1}^{n} \varepsilon_\mu(\sigma, t) \quad (\mu = 1, 2, \ldots, n).$$

When using τ_μ in considering the retardation time ($\tau_\mu = \eta_\mu / E_\mu$) and assuming the constant parameters of the Kelvin bodies (η_μ = const., E_μ = const. and thus also τ_μ = const.) within a time interval Δt_i, the strain increment due to concrete creep $\Delta\varepsilon_{c_i}$ the linear distribution of stresses in the time interval can be expressed by next equation:

$$\Delta\varepsilon_{c_i} = g(\bar{\sigma}) \sum_{\mu=1}^{n} \left\{ \left[\frac{\sigma_{i-1}}{E_{\mu_{i-1/2}}} - \varepsilon_\mu(t_{i-1}) \right] \frac{\lambda_{\mu_i} \cdot \Delta t_i}{\tau_\mu} + \frac{\Delta\sigma_i}{E_{\mu_{i-1/2}}} \cdot \left(1 - \lambda_{\mu_i}\right) \right\} \tag{30}$$

The first summand in the curly brackets of Eq. (30) represents the influence of the history of previous loading, and the second one the influence of the stress change in the current interval on the concrete creep. The advantage of such formulation of the concrete creep law is mainly caused by the fact that the entire history of previous loadings is included through the strains of the Kelvin bodies $\varepsilon_\mu(t_{i-1})$ and stress σ_{i-1} at the end of the previous time interval. Thus, in the case of an equilibrium state of a structure at time t_i, the strains $\varepsilon_\mu(t_i)$ of all Kelvin bodies in the chain, needed for the next time interval, must also be calculated:

$$\varepsilon_\mu(t_i) = \varepsilon_\mu(t_{i-1}) + g(\bar{\sigma}) \left\{ \left[\frac{\sigma_{i-1}}{E_{\mu_{i-1/2}}} - \varepsilon_\mu(t_{i-1}) \right] \frac{\lambda_{\mu_i} \cdot \Delta t_i}{\tau_\mu} + \frac{\Delta\sigma_i}{E_{\mu_{i-1/2}}} \cdot \left(1 - \lambda_{\mu_i}\right) \right\} \quad \mu = 1, 2, ., n \tag{31}$$

In Eqs. (30) and (31), the same functions as in the case of normal conditions during the increase in strength due to ageing, given in Eq. (25) to (27), are considered for the function $g(\bar{\sigma})$, which takes into account the influence of non-linear dependence of the creep strain on the stress level $\bar{\sigma} = |\sigma| / f_{cm\beta}(t)$. The concrete strength $f_{cm\beta}(t)$, which takes into account the decomposition due to long lasting high stress levels, is given in Eq. (15).

In the case of concrete decomposition, the stress-independent increment of strain due to concrete shrinkage $\Delta\varepsilon_{s_i}(\Delta t_i)$ is also considered according to Eq. (28).

3.2.3 Defining parameters of elements of generalised kelvin model of material

In the formulation of the constitutive law of concrete regarding the modelling of non-linear behaviour of concrete under short-term loading, the creep and shrinkage, the recommendations of the CEB-FIP Model Code 1990 are considered. In this way, the parameters of Kelvin bodies can be defined as follows [15]. The function, which describes the development of creep $\beta_c(t - t_0, \beta_H)$ (Eq. 2.1-70 in CEB-FIP Model Code 1990 [3]), is developed into Dirichlet series $\sum A_\mu(\beta_H) \cdot \left(1 - e^{-t/\tau_\mu}\right)$ [7]. Retardation times τ_μ are chosen in advance, and coefficients $A_\mu(\beta_H)$ are determined according to the least square method. By means of this method, polynomials of relatively low degree can be prepared in advance, which allows sufficiently accurate approximation of the coefficients $A_\mu(\beta_H)$ in the range of the expected values β_H. The next step is to define the parameters of Kelvin units according to Eq. (32). Quantities ϕ_{RH} , $\beta(f_{cm})$ and E_{ci} are determined by Eqs. (2.1-66), (2.1-67) and (2.1-16) in CEB-FIP Model Code 1990 [3]:

$$E_{\mu}(t) = E_{ci} \cdot \left(0{,}1 + t^{0{,}2}\right) \Big/ \left[\phi_{RH} \cdot \beta(f_{cm}) \cdot A_{\mu}(\beta_H)\right] \tag{32}$$

The main advantage of the chosen incremental constitutive law of concrete is that only the stress, the situation indicator (possible tensile or compressive failure) and the hidden variables of all Kelvin bodies in the previous time period have to be saved for each integrating point. This allows an analysis of large structures with an almost unlimited number of time steps to be performed by using merely a PC.

4. ANALYSIS OF STRUCTURES

Geometrical non-linearity of structures, material non-linearity including rheology and decomposition of concrete at high level of stresses are taken into account in the presented non-linear analysis of time dependent response of structures. The developed calculation method is based on the finite element method. Only the influence of normal stresses on displacements is taken into consideration, whereas the influence of shear stresses is neglected. The calculation of displacement and internal forces takes into account large displacements and moderate strains. The analysis of the structure is performed step by step at given discrete times t_i ($i = 1, 2, \dots m$).

The geometrical non-linearity of the structure is included by appropriate kinematic equations of a finite element (see Figure 3 [8]).

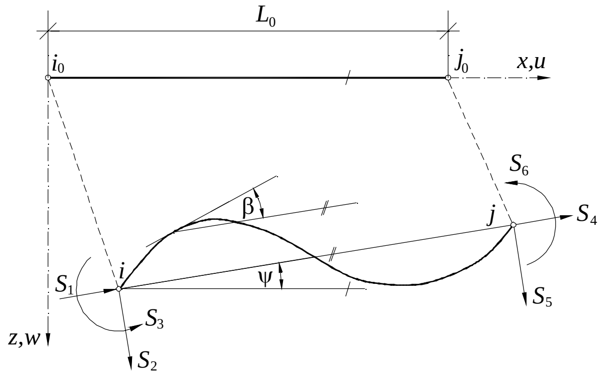

Fig. 3 Kinematics of a finite element

The elongation e_0 and the local declination β of direction of the deformed element axis from the connecting line between the nodes of the finite element can be expressed by:

$$\begin{aligned} (1+e_0)\cos\beta &= u_{,x} + \cos\psi \\ -(1+e_0)\sin\beta &= w_{,x} + \sin\psi \end{aligned} \tag{33}$$

where $u_{,x}$ and $w_{,x}$ are the derivatives of displacements u and w in the direction of x or z-axis with respect to the coordinate x, while ψ is the angle encompassed by the connecting line between the nodes of the deformed finite element and the initial direction of the element.

Bernoulli's hypothesis is adopted for the course of normal strains across the cross-section. In this way, the strain $\varepsilon(z,t)$ in time t can be expressed as $\varepsilon(z,t) = e_0(t) + z \cdot \beta_{,x}(t)$ where $e_0(t)$ represents the elongation, and $\beta_{,x}(t) = d\beta(t)/dx$ the curvature of the axis of the finite element [16]. If the time at the end of *i-th* interval t_i is taken for the time t and if it is considered that the geometrical strain $\varepsilon_{i-1}(z)$ in the initial time t_{i-1} is known, the increment of the geometrical strain $\Delta\varepsilon_i(z)$ in the time interval $\Delta t_i = t_i - t_{i-1}$ can be written by equation:

$$\begin{aligned} \Delta\varepsilon_i(z) &= \varepsilon_i(z) - \varepsilon_{i-1}(z) = \\ &= \left[e_0(t_i) + z\beta_{,x}(t_i)\right] - \varepsilon_{i-1}(z) \end{aligned} \tag{34}$$

where z is the coordinate of an arbitrary point of the cross-section.

In the analysis, the strain approach is used, where the change of stress $\Delta\sigma_i(z)$ for the known increment of strain $\Delta\varepsilon_i(z)$ is determined. With any known increment of the geometrical strain $\Delta\varepsilon_i(z)$, Eq. (34), in the *i-th* time interval, which must be identical to the corresponding increment of constitutive strain (Eq. (19)), the stress increment $\Delta\sigma_i(z)$ in an arbitrary point of the cross-section can be determined iteratively by an improved secant method solving the equation of form $f(\Delta\sigma_i)=0$. The function $f(\Delta\sigma_i)=0$ represents the difference between geometrical and constitutive strain. The stress at the end of the time interval $\sigma_i(z)$ concerned is defined by next equation:

$$\sigma_i(z) = \sigma(t_i, z) = \sigma_{i-1}(z) + \Delta\sigma_i(z) \tag{35}$$

The axial force $N(t_i)$ and the bending moment $M(t_i)$ of the cross-section in time t_i are obtained by integrating the stresses $\sigma(t_i, z)$ over the cross-section:

$$\begin{aligned} N(t_i) &= \int \sigma(t_i, z, e_0, \beta_{,x})\, dA \\ M(t_i) &= \int z \cdot \sigma(t_i, z, e_0, \beta_{,x})\, dA \end{aligned} \tag{36}$$

The equilibrium and kinematic conditions of individual nodes are taken into consideration when joining the elements into the structure. Since the equations of the structure are non-linear, they are solved iteratively step by step using the Newton-Raphson method. The finite element used is of a "mixed" type [8]. With this element, the curvature and axial force are approximated along its reference axis. On the assumption of the equilibrium as obtained at the beginning of the time interval t_{i-1}, the analysis of the stress-strain status of the structure at the end of the interval t_i concerned is performed in the following way. At any iteration, the curvature $\beta_{,x}$ and the angle ψ are determined by means of kinematic equations of the element. The elongation e_0 and the corresponding bending moment M for any integration node of the finite element, on the other hand, are determined on the basis of the constitutive equations of the cross-section, Eqs. (36). This is followed by the calculation of generalised forces and generalised elongations, as well as the calculation of the tangential stiffness matrix of the finite element. The tangential stiffness matrix of the structure is obtained by appropriately joining the matrices of individual finite elements. The equations of the structure in their incremental form are expressed for the *k-th* iteration in time t_i by the matrix equation:

$$\left[\bar{K}(t_i)\right]_{(k)}\{\Delta v_r\}_{(k+1)}=\{F(t_i)\}-\{\bar{S}_r(t_i)\}_{(k)} \qquad (37)$$

where $\left[\bar{K}(t_i)\right]_{(k)}$ is the tangential stiffness matrix of the structure, $\{\Delta v_r\}_{(k+1)}$ a vector of the unknown generalised node displacements, $\{F(t_i)\}$ the vector of a given node load and $\{\bar{S}_r(t_i)\}_{(k)}$ the vector of the condensed boundary forces of the finite element.

The iterations are repeated until the degree of precision required to meet the equilibrium conditions is obtained.

5. COMPUTATIONAL EXAMPLE

Benchmark test for evaluation of creep and shrinkage analysis computer programs, proposed by Subcommittee 3 of RILEM TC 114 [17], is analyzed as computational example. More comprehensive data relating to the analyzed experimental test can be found in the reference material [17]. The tensile strength, shrinkage, and creep functions of concrete used in the analysis are derived from its compressive strength according to CEB-FIP Model Code 1990 recommendations [3]. The proposal from the CEB-FIP Model Code 1990 (Eq. (25)) was also considered for the function, which takes into account the influence of non-linear creep in dependence of the level of stress.

Example comprises short and long-term tests of three slender reinforced concrete columns marked II-1, II-2 and II-3. These tests provide an illustration of the creep buckling phenomenon. The structural system, finite element model, cross-section and lay-out of reinforcement in the typical cross-section are shown in Figure 4. During the experiment, the relative air humidity was *55%* and the temperature *21°C*. Mean compressive strength at the age of 28 days was *38.3 MPa*. An axial load P was applied at the free end of the columns with an eccentricity $e = 1.5\ cm$.

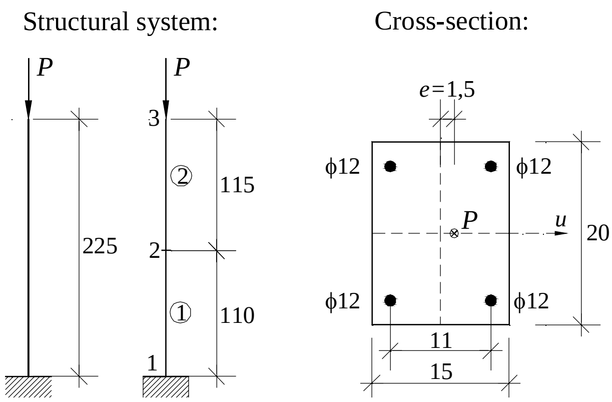

Fig. 4 Columns of benchmark example

The loading of specimens was performed in the following way:

* Column II-1 was loaded at the age of $t_0 = 28$ days with the monotonous increasing load until failure,
* Load $P = 280\ kN$ was applied to the column II-2 at $t_0 = 28$ days and sustained for 197 days when failure occurred,
* Column II-3 was loaded at the age of $t_0 = 28$ days by the force of $P=250\ kN$ and sustained for 206 days. Then the specimen was loaded for a short time until failure.

Figure 5 shows comparison between computationally and experimentally established relation between the force P and the horizontal displacement u in the node 3 of the specimen II-1. Calculations are made with function according to Eq. (25). The experimentally obtained failure force was $P_{u,1} = 444\ kN$, while the computationally determined value was $P^*_{u,1} = 432.5\ kN$, i.e. *97.4 %* of the experimentally obtained value. Global matching of the calculated and experimentally obtained *P-u* relations is also quite satisfactory.

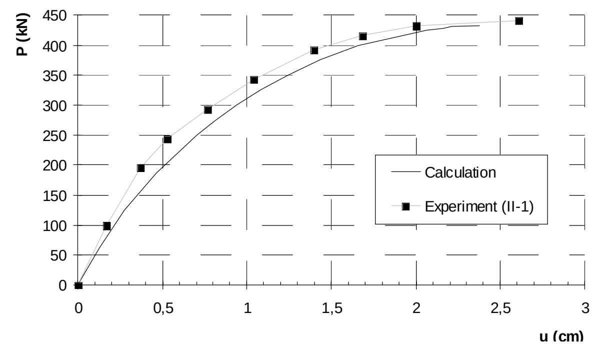

Fig. 5 Measured and calculated horizontal displacements in node 3 at monotonous loading of column II-1 until failure

Figure 6 shows comparison between measured and calculated horizontal displacement in the node 3 of the column II-2. The calculations are made using different functions $g(\bar{\sigma})$ (Eqs. (25), (26) and (27)) representing the influence of stress level on non-linear concrete creep. The specimen failed on the 197th day after the loading. The comparison between computational and experimental results shows considerably good matching of the predicted time of failure for the curves 1 and 3, while in the curve 2, which gives a better description of the displacement distribution in the initial period, the predicted failure time is significantly premature.

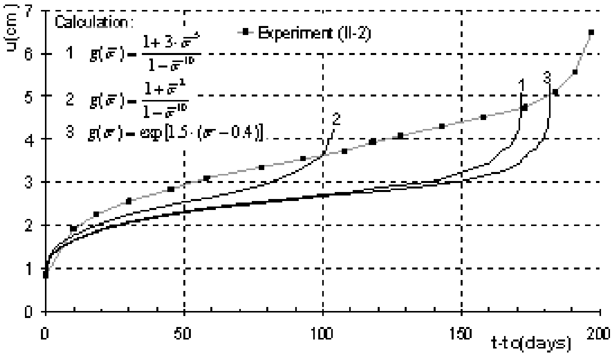

Fig. 6 . Measured and calculated time-dependent horizontal displacements in node 3 of column II-2

Figure 7 shows comparison between the computationally and experimentally determined time dependent courses of the horizontal displacements in node 3 of the specimen II-3 at constant loading of $P = 250\ kN$ sustained for 206 days.

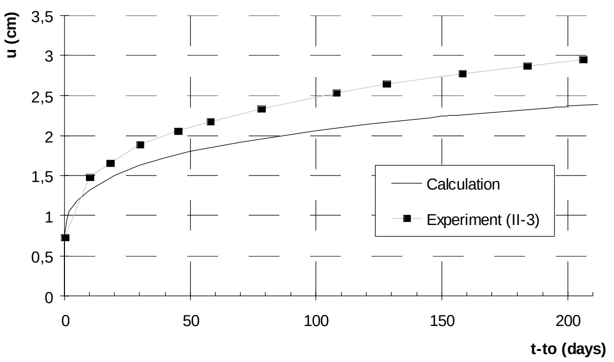

Fig. 7 Measured and calculated time-dependent horizontal displacements in node 3 of column II-3

Figure 8 shows the computationally and the experimentally determined relations between loading P and horizontal displacement of the top of the column for the same specimen during the initial loading of up to $P = 250\ kN$, the sustained loading of $P = 250\ kN$ for 206 days and the quick monotonous increase in loading until failure. The experimentally obtained failure force was $P_{u,3} = 331\ kN$. Using computational analysis, the failure force of $P^*_{u,3} = 325\ kN$ was determined, which equals *98.2%* of the experimentally obtained value. The increase of the displacement at the force $P = 250\ kN$ occurs due to rheological influences over the 206 days of constant loading.

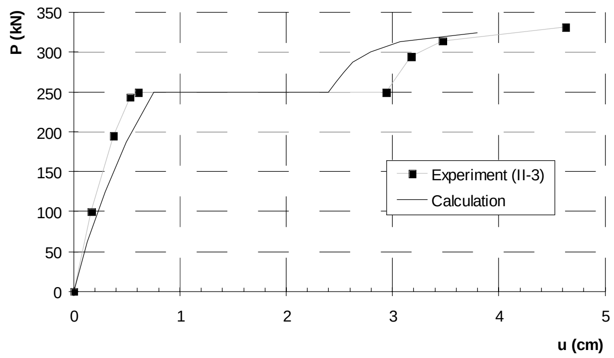

Fig. 8 Measured and calculated horizontal displacements in node 3 of column II-3

As well as experimentally established, the computational analysis also shows significant decrease in the bearing capacity of the specimen previously exposed to a constant load for a long time, as compared to short-term failed specimen. The experimentally obtained failure force for the specimen II-3 is only *74.5%* of the experimentally obtained value for the specimen II-1. Very close to this value is also the ratio between computational failure forces for both specimens $P^*_{u,3} / P^*_{u,1} = 75.1\ \%$. Here, the force of $P = 250\ kN$, which the sample II-3 was loaded with for 206 days until failure, is only *56.7 %* of the failure force for the short-term loading of specimen II-1.

6. CONCLUSION

At low stress level the concrete strength increases due to favourable influence of ageing, but on the other hand it decreases due to the decomposition of the concrete resulting from permanent action of high stresses. During ageing, the compressive strength of concrete increases by approximately up to *25%* with respect to the standard strength of the 28 days old concrete. As the consequence of decomposition of the concrete exposed to high stresses, the long-term strength of concrete decreases by about *15%* as compared to the short-term strength. Concrete strength $f_{cm\beta}(t)$ by taking into account the history of previous loading with high stress levels in arbitrary time t can be defined as the product of normalized strength $\beta_{cu}(t)$ and time-dependent concrete strength by taking into account only the influence of ageing $f_{cm}(t)$. Modelling of the basic mechanical non-linearity and the creep of concrete can be carried out by using the generalised Kelvin model of material consisting of a non-linear time-dependent spring and a chain of Kelvin bodies.

The developed method of analysis is limited to plane frames that can be reinforced, prestressed or composite. The level of stress can be arbitrary. The analysis is based on the finite element method. In addition to the geometrical non-linearity of the structure, the non-linearity of constitutive laws of material subject to short term loading and basic rheological influences, as well as the influence of high stress levels on the increase of concrete creep and the influence of sustained high stress levels on the reduction of compressive strength of concrete are considered. By taking into account the rheological influences according to the non-linear theory, the constitutive law of concrete is carried out in the incremental form. Based on the given theoretical derivations, the NONREO computer program for the non-linear analysis of time-dependent response of reinforced concrete plane frames was elaborated.

Comparing the results of the computer simulations of structure behaviour with foreign experiments, the developed analysis method and the software have proven to be adequate and useful in practice.

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NELINEARNA ANALIZA VREMENSKOG ODGOVORA BETONSKIH KONSTRUKCIJA

SAŽETAK

Ovaj rad se bavi mehaničkim i reološkim svojstvima betona, te numeričkim modeliranjem konstitutivnog zakona ponašanja betona uzimajući u obzir njegovo starenje, raspadanje i nelinearno puzanje.

Također iznosi metodu opće nelinearne analize vremenskog odgovora za betonske konstrukcije. Ograničava se na armirane, prednapregnute ili kompozitne ravninske okvire. Pored utjecaja reologije uključujući i nelinearno puzanje betona, može se uzeti u obzir vremenski promjenjiv konstrukcijski sustav. Stoga je ova metoda analize pogodna za predviđanje vremenskog odgovora konstrukcije (pomaci i unutarnje sile) kao i za procjenu sigurnosti betonskih konstrukcija u pojedinim fazama gradnje. Primjenjivost pripremljenog software-a kratko je prikazana primjerom.

Ključne riječi: *beton, reologija, starenje, skupljanje, puzanje, nelinearna anliza, vremenski odgovor, numeričko modeliranje, promjenjivi konstrukcijski sustav.*