UDC 531.391:519.61
Research note
Received: 07.10.2010.

New implicit method for analysis of problems in nonlinear structural dynamics

Ali Akbar Gholampour and **Mehdi Ghassemieh**

School of Civil Engineering, University of Tehran, Tehran, IRAN
e-mail: aagholampour@ut.ac.ir; mghassem@ut.ac.ir

SUMMARY

In this paper a new method is proposed for direct time integration of nonlinear structural dynamics problems. In the proposed method the order of time integration scheme is higher than the conventional Newmark's family of methods. This method assumes second order variation of the acceleration at each time step. Two variable parameters are used to increase the stability and accuracy of the method. The result obtained from this new higher order method is compared with two implicit methods; namely the Wilson-θ and the Newmark's average acceleration methods.

***Key words**: direct time integration, nonlinear structural dynamics, second order acceleration, implicit method.*

1. INTRODUCTION

Problems in the theory of vibration are divided in two categories: wave propagation problems and inertia problems. The latter is called structural dynamics. In these problems, governing field equation is a second order differential equation [1, 2]. For nonlinear systems, it is usually expected to solve numerically equations of motion [2]. In the time integration methods, time is divided into several time steps and an algorithm is used to predict the values of displacement, velocity or acceleration at each time based on previous value. The algorithm is based on an assumption for variation of displacement in each time step and in selected discontinuous times satisfying the equation of motion. In fact it is a form of finite difference solution for differential equations [2-11].

In nonlinear analysis, stiffness is calculated at the beginning of each time step and the response is calculated at the end of this time step assuming that stiffness is constant throughout the step. Therefore, nonlinearity is considered by continuously updating the stiffness. Calculated responses will be considered at the end of each time step as initial conditions for the next time step. Therefore, system nonlinearity behavior is replaced by a series of consecutive approximate linear characteristics [1, 2, 5, 8].

In some of algorithms, in each time step the equation of motion is written at the beginning of the time step and the unknown values at the end of time step are explicitly calculated. These methods are called explicit methods. In some other methods, if the unknown values are calculated at the end of time step it is required to write the equation of motion at this point. These methods are called implicit methods [2-9]. A method is called convergent if its error for a specific time is decreased by decreasing time step length. Also, a method is consistent if the upper bound of its residue (error in satisfying the equation of motion), is a constant power of time step length. In accuracy evaluation of the time integration methods, two quantities usually are determined, numerical damping (dissipation) and periodic error (dispersion).

Instability never happens in unconditionally stable methods, no matter how long time step is [1-5]. Newmark [12] presented one step algorithm with two parameters. He noted that γ should be taken as *0.5*, because for values more than *0.5* a positive numerical damping will exist and for values less than that, it will have negative numerical damping (numerical instability). The average acceleration form appears to be the most popular one. After him, many researches have worked on his idea. Wilson presented a modified form of linear acceleration method, called Wilson-θ method [13]. He improved it to be an unconditionally stable method. He also proposed the concept of collocation to develop dissipative algorithms, which were further generalized in Ref. [13]. The Wilson-θ method is unconditionally stable for $\theta=1.37$. This method is a subject to both phase and amplitude errors depending on the time step used.

Classical methods such as the Newmark's method [12] or the Wilson-θ method [13] assume a constant or linear expression for the variation of acceleration at each time step. In conditionally stable methods, the time step must be smaller than a critical time step as a constant times the smallest period of the system, consequently it often entails the use of time steps that are much smaller than those needed for accuracy [7]. In this paper, we illustrate how to derive equations of proposed method from the Taylor series expansion in which algorithmic parameters are inserted. In this new implicit method, it is assumed that the acceleration varies quadratically within each time step. The proposed method is derived by considering this assumption and by employing the two parameters δ and α.

2. PROPOSED METHOD

The governing nonlinear equation of motion is expressed as:

$$\boldsymbol{M}\,\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}_n\,\boldsymbol{x} = \boldsymbol{P} \tag{1}$$

where $\boldsymbol{M}$ and $\boldsymbol{C}$ are the mass and damping matrices; $\boldsymbol{K}_n$ is stiffness matrix in the *n-th* time step; $\boldsymbol{P}$ is vector of applied forces; $\boldsymbol{x}$, $\dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}}$ are the displacement, velocity and acceleration vectors respectively.

By applying the Taylor series expansions of $x_{t+\Delta t}$ and $\dot{x}_{t+\Delta t}$ about time t and truncating the equations, the following forms of equations are obtained:

$$x_{t+\Delta t} = x_t + \Delta t\,\dot{x}_t + \frac{\Delta t^2}{2}\ddot{x}_t + \frac{\Delta t^3}{6}\dddot{x}_t + \alpha\Delta t^4 \ddddot{x}_t \tag{2}$$

$$\dot{x}_{t+\Delta t} = \dot{x}_t + \Delta t\,\ddot{x}_t + \frac{\Delta t^2}{2}\dddot{x}_t + \delta\Delta t^3 \ddddot{x}_t \tag{3}$$

If the acceleration variation is assumed to be second order within time t-Δt to $t+\Delta t$, the Eqs. (2) and (3) can be written as:

$$x_{t+\Delta t} = x_t + \Delta t\,\dot{x}_t + \left[\left(\alpha - \frac{1}{12}\right)\ddot{x}_{t-\Delta t} + \left(\frac{1}{2} - 2\alpha\right)\ddot{x}_t + \left(\alpha + \frac{1}{12}\right)\ddot{x}_{t+\Delta t}\right]\Delta t^2 \tag{4}$$

$$\dot{x}_{t+\Delta t} = \dot{x}_t + \left[\left(\delta - \frac{1}{4}\right)\ddot{x}_{t-\Delta t} + (1-2\delta)\ddot{x}_t + \left(\delta + \frac{1}{4}\right)\ddot{x}_{t+\Delta t}\right]\Delta t \tag{5}$$

Equations (4) and (5) can be used to approximate the displacement and velocity at time $t+\Delta t$ respectively. It can be proven that this strategy guarantees the second-order accuracy for any choice of δ and α. The parameters δ and α are introduced in order to improve accuracy and stability. Special case $\delta=1/4$ and $\alpha=1/2$ leads to the linear acceleration method.

Consider equation of motion in time $t+\Delta t$ as following:

$$\boldsymbol{M}\,\ddot{x}_{t+\Delta t} + \boldsymbol{C}\,\dot{x}_{t+\Delta t} + \boldsymbol{K}_t\,x_{t+\Delta t} = \boldsymbol{P}_{t+\Delta t} \tag{6}$$

By substituting Eqs. (4) and (5) into the equation of motion, Eq. (6), $\ddot{x}_{t+\Delta t}$ is calculated. Note that x_0 and $\dot{x}_0$ are known and can be calculated using Eq. (1) at time $t=0$. We need the solution at time before we can begin to apply Eqs. (4) and (5). It can be computed by using any of one step methods such as the linear acceleration or the average acceleration methods. Now, we can obtain $x_{2\Delta t}$ from Eqs. (4) and (5), then $x_{3\Delta t}$, and so on.

3. EXAMPLES

In order to see the result of the proposed method and its advantages over the other implicit existing methods, let's consider two examples the results of which obtained from the proposed method are compared with the Wilson-θ and average acceleration (Newmark's) methods.

Example 1 [2]: Consider a single degree of freedom system shown in Figure 1 with elastoplastic behavior shown in Figure 2 under exciting force being applied on the spring damping system shown in Figure 3.

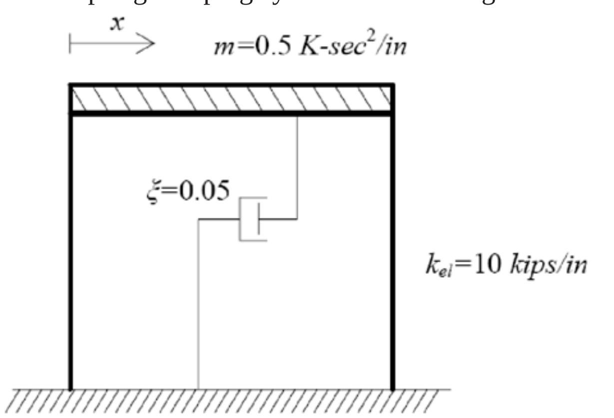

Fig. 1 Frame of structure [2]

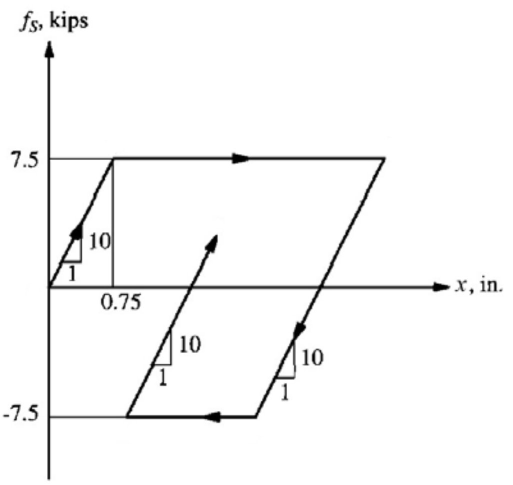

Fig. 2 Force-displacement relationship [2]

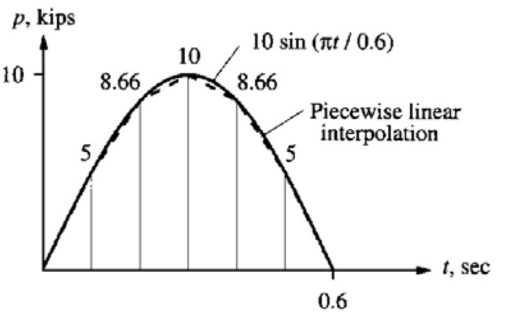

Fig. 3 Exciting force [2]

In Table 1 displacement results of this system due to the applied loading $P(t)$ (see Figure 3) are given. The proposed methods are compared by the obtained results which use Wilson-θ and average (Newmark's) acceleration methods. The displacement responses versus time are also shown in Figure 4.

Table 1. Numerical responses using the Wilson-θ, average acceleration and proposed methods

Time (sec)	*Displacement Responses (in)*		
	Wilson-θ *(θ=1.4)*	***Average acceleration method***	***Proposed method*** *(δ=1/3, α=1/6)*
0	*0*	*0*	*0*
0.1	*0.0368*	*0.0437*	*0.0437*
0.2	*0.1833*	*0.2326*	*0.2195*
0.3	*0.4830*	*0.6121*	*0.5909*
0.4	*0.9007*	*1.0825*	*1.0616*
0.5	*1.3226*	*1.5279*	*1.4822*
0.6	*1.6828*	*1.8377*	*1.7394*
0.7	*1.9783*	*1.8893*	*1.7422*
0.8	*1.8623*	*1.6716*	*1.4826*
0.9	*1.3011*	*1.2801*	*1.0656*

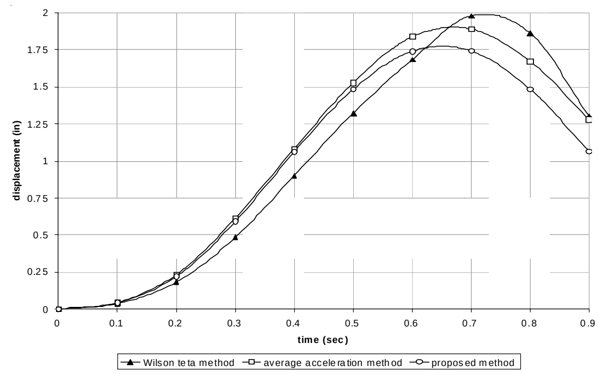

Fig. 4 Displacement responses versus time diagram for example 1

Example 2 [14]: Consider the second order nonlinear differential equation as followsg:

$$\ddot{x} + \sin x = 0 \tag{7}$$

with initial conditions $x(0)=\pi/2$ and $\dot{x}(0)=0$ that $0 \le t \le 20$. Let's select $\Delta t = 0.1$ and define the error at time t as follows:

$$e_t = \left| x_t - x_{t(exact)} \right| \tag{8}$$

in which $x_{t(exact)}$ is the exact solution and x_t is the numerical solution (angle (degree)) at time t. The proposed methods can be compared with each other in Figure 5 by using the values obtained by the Wilson-θ and average (Newmark's) acceleration methods.

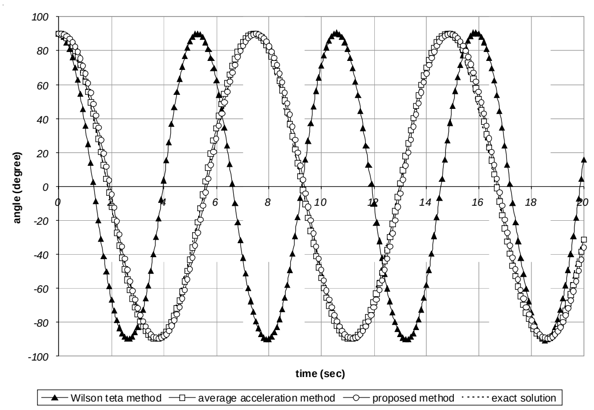

Fig. 5 Angle responses versus time diagram for example 2

The numerical solution calculated by the mentioned methods and their error with respect to the exact solution of Eq. (7) has been shown in Table 2 for $t=6$ *sec* to $t=7$ *sec*. Table 2 shows that the numerical values of x_t calculated using the proposed method are more accurate than those for the Wilson-θ and average acceleration methods. In this example, we presented only angle responses, whereas the angular velocity and angular acceleration responses calculated using the proposed method are also more accurate than the other methods.

Table 2. Angle responses using the Wilson-θ, average acceleration and proposed methods and their error respect to the exact solution

Time (sec)	Wilson - θ method (θ =1.4)		Average acceleration method		Proposed method (δ =1/3, α =1/6)	
	x_t	e_t	x_t	e_t	x_t	e_t
6	62.1602	33.3314	34.3660	5.5372	28.1380	0.6908
6.1	53.8810	18.2972	41.5566	5.9728	35.5922	0.0084
6.2	44.6907	1.9390	48.3691	5.6174	42.7140	0.0376
6.3	34.7041	14.5329	54.7518	5.5149	49.4463	0.2093
6.4	24.0700	31.2963	60.6705	5.3042	55.7431	0.3768
6.5	12.9718	47.4454	66.0907	5.6736	61.5644	1.1472
6.6	1.6215	64.6319	70.9838	4.7304	66.8814	0.6280
6.7	-9.7689	81.3105	75.3383	3.7967	71.6713	0.1298
6.8	-20.9645	96.2283	79.1427	3.8789	75.9170	0.6531
6.9	-31.7591	111.718	82.3799	2.4214	79.6068	0.3517
7	-41.9635	125.158	85.0499	1.8557	82.7294	0.4647

4. CONCLUSIONS

A new implicit step by step integration technique for the problems in structural dynamics was illustrated. A second order polynomial was used as a function of time in order to approximate the variation of acceleration during the time steps. Therefore, the proposed method showed more accurate values than the Wilson-θ and average acceleration methods. This method was a two parameter method (δ and α). The proposed method allows numerical damping while it retains a second order accuracy. The new method can be used for either linear or nonlinear problems, though in this paper, we have discussed only nonlinear problems.

5. REFERENCES

[1] M. Paz, *Structural Dynamics: Theory and Computation*, 4th ed., Chapman & Hall, New York, 1997.

[2] A. Chopra, *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, 3rd ed., Prentice-Hall, Upper Saddle River, New Jersey, 2007.

[3] T.J.R. Hughes and T. Belytschko, A precis of developments in computational methods for transient analysis, *Journal of Applied Mechanics*, Vol. 50, No. 4b, pp. 1033-1041, 1983.

[4] K. Subbaraj and M.A. Dokainish, A survey of direct time integration methods in computational structural dynamics – Part II. Implicit methods, *Computers & Structures*, Vol. 32, No. 6, pp. 1387-1401, 1989.

[5] J.L. Humar, *Dynamics of Structures*, Prentice-Hall, Englewood Cliffs, New Jersey, 1990.

[6] K.J. Bathe, *Finite Element Procedures*, Prentice-Hall, Englewood Cliffs, New Jersey, 1996.

[7] M.A. Crisfield, *Non-Linear Finite Element Analysis of Solids and Structures*, Vol. 2, John Wiley & Sons, 1997.

[8] R.W. Clough and J. Penzien, *Dynamics of Structures*, McGraw Hill, 1983.

[9] K.C. Park, Practical aspects of numerical time integration, *Computers & Structures*, Vol. 7, No. 3, pp. 343-353, 1977.

[10] T. Belytschko, W.K. Liu and B. Moran, *Nonlinear Finite Elements for Continua and Structures*, 3rd ed., John Wiley & Sons, Chichester, 2000.

[11] Sh. Chen, J.M. Hansen and D.A. Tortorelli, Unconditionally energy stable implicit time integration: Application to multibody system analysis and design, *International Journal for Numerical Methods in Engineering*, Vol. 48, No. 6, pp. 791-822, 2000.

[12] N.M. Newmark, A method of computation for structural dynamics, *Journal of the Engineering Mechanics Division*, Vol. 85, No. 3, pp. 67-94, 1959.

[13] E.L. Wilson, I. Farhoomand and K.J. Bathe, Nonlinear Dynamic Analysis of Complex Structures, *Int. J. of Earthquake Engineering and Structural Dynamics*, Vol. 1, No. 3, pp. 241–252, 1973.

[14] C.W. Bert and J.D. Striklin, Comparative evaluation of six different numerical integration methods for non-linear dynamic systems, *Journal of Sound and Vibration*, Vol. 127, No. 2, pp. 221-229, 1988.

NOTATIONS

t - time
D_t - time step
M - mass matrix
C - damping matrix
K_n - stiffness matrix at nth time step
P - applied force vector
x - displacement vector
$\dot{x}$ - velocity vector
$\ddot{x}$ - acceleration vector
x_t - displacement vector at time t
$x_{t(exact)}$ - exact value of the displacement vector at time t
x - damping ratio
e_t - error at time t
f_s - spring force

NOVA IMPLICITNA METODA ZA ANALIZU PROBLEMA U NELINEARNOJ DINAMIČKOJ ANALIZI KONSTRUKCIJA

SAŽETAK

Ovaj rad predlaže novu metodu za direktnu vremensku integraciju u nelinearnoj dinamičkoj analizi konstrukcija. U predloženom modelu red vremenske integracije je viši nego kod Newmark-ovih metoda. Ova metoda pretpostavlja varijaciju akceleracije drugoga reda u svakom pojedinom vremenskom periodu. Koriste se dva parametra varijable kako bi se povećala stabilnost i točnost ove metode. Dobiveni rezultat pomoću ove nove metode višega reda uspoređuje se s dvije implicitne metode; s Wilson-ovom θ i Newmark-ovom metodom srednje vrijednosti.

Ključne riječi: *integracija direktnoga vremena, nelinearna dinamička analiza, akceleracija drugog reda, implicitna metoda.*