New implicit method for analysis of problems in nonlinear structural dynamics

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SUMMARY

In this paper a new method is proposed for direct time integration of nonlinear structural dynamics problems. In the proposed method the order of time integration scheme is higher than the conventional Newmark’s family of methods. This method assumes second order variation of the acceleration at each time step. Two variable parameters are used to increase the stability and accuracy of the method. The result obtained from this new higher order method is compared with two implicit methods; namely the Wilson-\(\theta\) and the Newmark’s average acceleration methods.

Key words: direct time integration, nonlinear structural dynamics, second order acceleration, implicit method.

1. INTRODUCTION

Problems in the theory of vibration are divided in two categories: wave propagation problems and inertia problems. The latter is called structural dynamics. In these problems, governing field equation is a second order differential equation [1, 2]. For nonlinear systems, it is usually expected to solve numerically equations of motion [2]. In the time integration methods, time is divided into several time steps and an algorithm is used to predict the values of displacement, velocity or acceleration at each time based on previous value. The algorithm is based on an assumption for variation of displacement in each time step and in selected discontinuous times satisfying the equation of motion. In fact it is a form of finite difference solution for differential equations [2-11].

In nonlinear analysis, stiffness is calculated at the beginning of each time step and the response is calculated at the end of this time step assuming that stiffness is constant throughout the step. Therefore, nonlinearity is considered by continuously updating the stiffness. Calculated responses will be considered at the end of each time step as initial conditions for the next time step. Therefore, system nonlinearity behavior is replaced by a series of consecutive approximate linear characteristics [1, 2, 5, 8].

In some of algorithms, in each time step the equation of motion is written at the beginning of the time step and the unknown values at the end of time step are explicitly calculated. These methods are called explicit methods. In some other methods, if the unknown values are calculated at the end of time step it is required to write the equation of motion at this point. These methods are called implicit methods [2-9]. A method is called convergent if its error for a specific time is decreased by decreasing time step length. Also, a method is consistent if the upper bound of its residue (error in satisfying the equation of motion), is a constant power of time step length. In accuracy evaluation of the time integration methods, two quantities usually are determined, numerical damping (dissipation) and periodic error (dispersion).
Instability never happens in unconditionally stable methods, no matter how long time step is [1-5]. Newmark [12] presented one step algorithm with two parameters. He noted that γ should be taken as 0.5, because for values more than 0.5 a positive numerical damping will exist and for values less than that, it will have negative numerical damping (numerical instability). The average acceleration form appears to be the most popular one. After him, many researches have worked on his idea. Wilson presented a modified form of linear acceleration method, called Wilson-θ method [13]. He improved it to be an unconditionally stable method. He also proposed the concept of collocation to develop dissipative algorithms, which were further generalized in Ref. [13]. The Wilson-θ method is unconditionally stable for θ=1.37. This method is a subject to both phase and amplitude errors depending on the time step used.

Classical methods such as the Newmark’s method [12] or the Wilson-θ method [13] assume a constant or linear expression for the variation of acceleration at each time step. In conditionally stable methods, the time step must be smaller than a critical time step as a constant times the smallest period of the system, consequently it often entails the use of time steps that are much smaller than those needed for accuracy [7]. In this paper, we illustrate how to derive equations of proposed method from the Taylor series expansion in which algorithmic parameters are inserted. In this new implicit method, it is assumed that the acceleration varies quadratically within each time step. The proposed method is derived by considering this assumption and by employing the two parameters δ and α.

2. PROPOSED METHOD

The governing nonlinear equation of motion is expressed as:

\[ M \ddot{x} + C \dot{x} + K_n x = P \]  

(1)

where \( M \) and \( C \) are the mass and damping matrices; \( K_n \) is stiffness matrix in the \( n \)-th time step; \( P \) is vector of applied forces; \( x, \dot{x} \) and \( \ddot{x} \) are the displacement, velocity and acceleration vectors respectively.

By applying the Taylor series expansions of \( x_{t+\Delta t} \) and \( \dot{x}_{t+\Delta t} \) about time \( t \) and truncating the equations, the following forms of equations are obtained:

\[ x_{t+\Delta t} = x_t + \Delta t \dot{x}_t + \frac{\Delta t^2}{2} \ddot{x}_t + \frac{\Delta t^3}{6} \dddot{x}_t + \alpha \Delta t^4 \dddot{x}_t \]  

(2)

\[ \ddot{x}_{t+\Delta t} = \ddot{x}_t + \Delta t \dddot{x}_t + \frac{\Delta t^2}{2} \dddot{x}_t + \delta \Delta t^3 \dddot{x}_t \]  

(3)

If the acceleration variation is assumed to be second order within time \( t-\Delta t \) to \( t+\Delta t \), the Eqs. (2) and (3) can be written as:

\[ \begin{align*}
  x_{t+\Delta t} &= x_t + \Delta t \dot{x}_t + \\
                   &+ \left[ \left( \alpha - \frac{1}{12} \right) \ddot{x}_{t-\Delta t} + \left( \frac{1}{2} - 2\alpha \right) \ddot{x}_t + \left( \alpha + \frac{1}{12} \right) \ddot{x}_{t+\Delta t} \right] \Delta t^2
\end{align*} \]  

(4)

\[ \begin{align*}
  \ddot{x}_{t+\Delta t} &= \ddot{x}_t + \\
                    &+ \left[ \left( \delta - \frac{1}{4} \right) \dddot{x}_{t-\Delta t} + \left( 1 - 2\delta \right) \dddot{x}_t + \left( \delta + \frac{1}{4} \right) \dddot{x}_{t+\Delta t} \right] \Delta t
\end{align*} \]  

(5)

Equations (4) and (5) can be used to approximate the displacement and velocity at time \( t+\Delta t \) respectively. It can be proven that this strategy guarantees the second-order accuracy for any choice of δ and α. The parameters δ and α are introduced in order to improve accuracy and stability. Special case δ=1/4 and α=1/2 leads to the linear acceleration method.

Consider equation of motion in time \( t+\Delta t \) as following:

\[ M \ddot{x}_{t+\Delta t} + C \dot{x}_{t+\Delta t} + K_n x_{t+\Delta t} = P_{t+\Delta t} \]  

(6)

By substituting Eqs. (4) and (5) into the equation of motion, Eq. (6), \( \ddot{x}_{t+\Delta t} \) is calculated. Note that \( x_0 \) and \( \dot{x}_0 \) are known and can be calculated using Eq. (1) at time \( t=0 \). We need the solution at time before we can begin to apply Eqs. (4) and (5). It can be computed by using any of one step methods such as the linear acceleration or the average acceleration methods. Now, we can obtain \( x_{2\Delta t} \) from Eqs. (4) and (5), then \( x_{3\Delta t} \) and so on.

3. EXAMPLES

In order to see the result of the proposed method and its advantages over the other implicit existing methods, let’s consider two examples the results of which obtained from the proposed method are compared with the Wilson-θ and average acceleration (Newmark’s) methods.

**Example 1** [2]: Consider a single degree of freedom system shown in Figure 1 with elastoplastic behavior shown in Figure 2 under exciting force being applied on the spring damping system shown in Figure 3.

![Example 1](image)

**Fig. 1 Frame of structure [2]**
Consider the second order nonlinear differential equation as follow:

\[ \ddot{x} + \sin x = 0 \]  

with initial conditions \( x(0) = \pi/2 \) and \( \dot{x}(0) = 0 \) that \( 0 \leq t \leq 20 \). Let's select \( \Delta t = 0.1 \) and define the error at time \( t \) as follows:

\[ e_t = |x_t - x_{t(\text{exact})}| \]  

in which \( x_{t(\text{exact})} \) is the exact solution and \( x_t \) is the numerical solution (angle (degree)) at time \( t \). The proposed methods can be compared with each other in Figure 5 by using the values obtained by the Wilson-\( \theta \) and average (Newmark's) acceleration methods.

Table 1. Numerical responses using the Wilson-\( \theta \), average acceleration and proposed methods

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>Wilson-( \theta ) (( \theta = 1.4 ))</th>
<th>Average acceleration method</th>
<th>Proposed method (( \delta = 1/3, \alpha = 1/6 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0368</td>
<td>0.0437</td>
<td>0.0437</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1833</td>
<td>0.2326</td>
<td>0.2195</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4830</td>
<td>0.6121</td>
<td>0.5909</td>
</tr>
<tr>
<td>0.4</td>
<td>0.9007</td>
<td>1.0825</td>
<td>1.0616</td>
</tr>
<tr>
<td>0.5</td>
<td>1.3226</td>
<td>1.5279</td>
<td>1.4822</td>
</tr>
<tr>
<td>0.6</td>
<td>1.6828</td>
<td>1.8377</td>
<td>1.7394</td>
</tr>
<tr>
<td>0.7</td>
<td>1.9783</td>
<td>1.8893</td>
<td>1.7422</td>
</tr>
<tr>
<td>0.8</td>
<td>1.8623</td>
<td>1.6716</td>
<td>1.4826</td>
</tr>
<tr>
<td>0.9</td>
<td>1.3011</td>
<td>1.2801</td>
<td>1.0656</td>
</tr>
</tbody>
</table>

The numerical solution calculated by the mentioned methods and their error with respect to the exact solution of Eq. (7) has been shown in Table 2 for \( t = 6 \) sec to \( t = 7 \) sec. Table 2 shows that the numerical values of \( x_t \) calculated using the proposed method are more accurate than those for the Wilson-\( \theta \) and average acceleration methods. In this example, we presented only angle responses, whereas the angular velocity and angular acceleration responses calculated using the proposed method are also more accurate than the other methods.
4. CONCLUSIONS

A new implicit step by step integration technique for the problems in structural dynamics was illustrated. A second order polynomial was used as a function of time in order to approximate the variation of acceleration during the time steps. Therefore, the proposed method showed more accurate values than the Wilson-θ and average acceleration methods. This method was a two parameter method \((\delta, \alpha)\). The proposed method allows numerical damping while it retains a second order accuracy. The new method can be used for either linear or nonlinear problems, though in this paper, we have discussed only nonlinear problems.

5. REFERENCES


A.A. Gholampour, M. Ghassemieh: New implicit method for analysis of problems in nonlinear structural dynamics

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NOVA IMPLICITNA METODA ZA ANALIZU PROBLEMA U NELINEARNOJ DINAMIČKOJ ANALIZI KONSTRUKCIJA

SAŽETAK

Ovaj rad predlaže novu metodu za direktnu vremensku integraciju u nelinearnoj dinamičkoj analizi konstrukcija. U predloženom modelu red vremenske integracije je viši nego kod Newmark-ovih metoda. Ova metoda pretpostavlja varijaciju akceleracije drugoga reda u svakom pojedinom vremenskom periodu. Koriste se dva parametra varijable kako bi se povećala stabilnost i točnost ove metode. Dobiveni rezultat pomoću ove nove metode višega reda uspoređuje se s dvije implicitne metode; s Wilson-ovom \( \theta \) i Newmark-ovom metodom srednje vrijednosti.

Ključne riječi: integracija direktnoga vremena, nelinearna dinamička analiza, akceleracija drugog reda, implicitna metoda.


NOTATIONS

\( t \) - time
\( D_t \) - time step
\( M \) - mass matrix
\( C \) - damping matrix
\( K_n \) - stiffness matrix at nth time step
\( P \) - applied force vector
\( x \) - displacement vector
\( \dot{x} \) - velocity vector
\( \ddot{x} \) - acceleration vector
\( x_t \) - displacement vector at time \( t \)
\( x_t^{(exact)} \) - exact value of the displacement vector at time \( t \)
\( \chi \) - damping ratio
\( e_t \) - error at time \( t \)
\( f_s \) - spring force