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# An EPQ model with a random breakdown, rework, and a discontinuous issuing policy

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## SUMMARY

*This study attempts to determine the optimal production run time for an economic production quantity (EPQ) model with a Poisson breakdown rate, the rework of nonconforming items, and multi-delivery of finished products. It is assumed that nonconforming items can be randomly produced and repaired through a rework process added at the end of the production process. Production equipment is subject to a random breakdown, which follows a Poisson distribution. When a breakdown occurs, the machine immediately goes under repair and adopts an abort/resume inventory control policy. Under this policy, the production of the interrupted lot instantaneously resumes when the machine is repaired and restored. Upon completion of the rework process, n fixed quantity installments of the finished batch are delivered to the customer at a fixed interval of time. Mathematical modelling together with a numerical analysis is used in this study. Theorems on the convexity of the expected system cost function and bounds of production uptime are proposed and proved. A recursive searching algorithm is developed to find the optimal replenishment uptime within the bounds. Finally, a numerical example is provided to demonstrate the practical usage of the study results.*

**KEY WORDS:** *economic production quantity, breakdown, multi-delivery, rework, abort-resume policy, optimization.*

## 1. INTRODUCTION

The economic production quantity (EPQ) model is often adopted by manufacturing firms to deal with the most economic, non-instantaneous replenishment issues [1-4]. A classic EPQ model assumes that all items produced are of perfect quality. However, in real-life production systems due to process deterioration or various uncontrollable factors, a generation of defective items is inevitable. Therefore, many studies have been carried out to enhance the classic EPQ model by addressing the issues of imperfections in the quality of products and quality assurance matters [5-12].

In addition to the defective rate, random breakdown is another inevitable critical reliability factor that can be very disruptive when it occurs, especially in a highly automated production environments. Groenevelt *et al.* [13] presented two production control policies to deal with the breakdown; first one suggests that the production of an interrupted lot is not resumed (called 'no resumption' or 'NR policy') after a breakdown; and the second policy considers that the production of an interrupted lot will be immediately resumed (called 'abort-resume' or 'AR policy') after a breakdown is fixed, and if the current on-hand inventory is below a certain threshold level. Kuhn [14] investigated a dynamic lot sizing model with exponential machine breakdowns. Two different situations were examined. In case study one, after a machine breakdown the setup is totally lost and new setup cost is incurred. In case study two, the cost of resuming the production run after a failure might be substantially lower than the production setup cost. The study showed that, in the first case, the cost penalty for ignoring machine failures would be noticeably higher than that of the classical EPQ model. For the second case, a conditional resumption, based on the sizes of future demands versus the incomplete lot sizes, was recommended. Kuhn also suggested a stochastic dynamic programming model for finding optimum lot sizing decisions for both cases. Makis and Fung [15] examined an economic manufacturing quantity (EMQ) model with inspections and random machine failures. Effects of breakdowns of an optimal lot size and optimal number of inspections were studied. The formula for the long-run expected average cost per unit time was obtained, and the optimal production and inspection policy that would minimize the expected average costs were derived. Giri and Dohi [16] presented the exact formulation of stochastic EMQ model for an unreliable production system. Their EMQ model was formulated on the basis of the net present value (NPV) approach and, by taking limitation on the discount rate, a traditional long-run, average cost model was obtained. The criteria for the existence and uniqueness of an optimal production time and its computational results were provided to show that the optimal decision based on the NPV approach is superior to that based on the long-run average cost approach. Chiu [17] examined the production run time problem with random machine breakdowns under abort/resume policy and reworking of defective items produced. Mathematical modelling and derivation of the production-inventory cost functions for both systems with/without breakdowns were presented. These functions were integrated and the long-run average cost per unit time was obtained. Theorems on convexity and on bounds of run time were proposed and proved. A recursive searching algorithm was developed for locating optimal run time. There are additional studies [18-23] that address different issues of production systems with breakdown.

Another unrealistic assumption of the classic EPQ model is 'continuous inventory issuing policy' for satisfying product demand. In real supply chains environments, it is common to adopt a multi-delivery policy for transporting finished items to customer. Schwarz [24] considered a one-warehouse, N-retailer inventory system to determine an optimal stocking policy that would minimize an average system cost. He derived some necessary properties for an optimal policy as well as the optimal solution. Heuristic solutions were also provided for the general problem and tested against analytical lower bounds. Goyal [25] studied integrated inventory model for a single supplier-single customer problem. Banerjee [26] examined a joint economic lot-size model for purchaser and vendor, with the focus on minimizing the joint total relevant cost. He concluded that a jointly optimal ordering policy, together with an appropriate price adjustment, could be economically beneficial for both parties, but definitely not disadvantageous to either party. Viswanathan [27] reexamined the integrated vendor-buyer inventory models with two different strategies: one where each replenishing quantity delivered to the buyer is identical and the other strategy where at each delivery all the inventory available with the vendor is supplied to the buyer. A detailed numerical analysis of the relative performance of the two strategies for various problem parameters is presented. Hoque [28] studied a model of delivering a single product to multiple buyers when the set-up and inventory costs to the vendor are included. A close relationship between a manufacturer and buyers is assumed for a costless way of benefit sharing. Three models were examined, two of which consider equal batches, and the third with unequal batches of the product. Optimal solution techniques were presented, sensitivity analyses of the techniques were carried out, and several numerical problems were solved to support the analytical findings. Chiu *et al*. [29] combined a multi-delivery policy and quality assurance into an imperfect EPQ model with scrap and rework. They assumed that random defective items produced are partially repairable and the finished items can only be delivered to customers if the entire lot is quality assured at the end of rework. The expected integrated cost function per unit time was derived, and a closed- form optimal batch size solution to the problem was obtained. Other studies that focused on various aspects of supply chain optimization can also be found [30-35].

This paper incorporates a multi-delivery policy into Chiu's model [17] and studies the joint effects of a multi-delivery policy, random breakdown, and rework on the optimal replenishment run time for a specific imperfect EPQ model.

## 2. PROBLEM DESCRIPTION AND MATHEMATICAL MODELLING

This study examines an imperfect EPQ-based model with a random breakdown, rework, and multi-delivery policy. The model assumes that a production process may randomly produce $x$ portion of defective items at a rate $d_1$. For a regular supply, the constant production rate $P$ is greater than a demand rate $\lambda$. Hence, $P_1$ has to be greater than the sum of $\lambda$ and $d_1$, i.e., $(P_1-d_1-\lambda)>0$ where $d_1=P_1x$. All items produced are screened and inspection cost is included in the unit production cost $c$. A defective rate $x$ is assumed to be a random variable with a known probability density function. All defective items are assumed to be repairable through a rework process at a rate $P_2$ in each production run, after the regular process has finished.

During the production uptime, a machine is subject to a random breakdown that follows the Poisson distribution. When a breakdown occurs, an AR policy is adopted, wherein the machine is put under repair immediately and the interrupted lot resumes when the machine is repaired and back in the operating condition. Machine repair time is assumed to be constant. Further, this paper incorporates a multi-delivery policy into the aforementioned imperfect EPQ model with rework and random breakdown. It is assumed that finished items can only be delivered to customer if the whole lot is quality assured, at the end of rework. A fixed quantity of $n$ installments of a finished batch is delivered at a fixed interval of time during production downtime $t_3$ (see Figure 1). Cost parameters considered in the proposed model include the production setup cost $K$, holding cost $h$, unit rework cost $C_R$, holding cost $h_1$ for each reworked item, fixed delivery cost $K_1$ per shipment, and a delivery cost $C_T$ per item shipped to the customer, and the cost for repairing and restoring a machine $M$. Additional notations used in this study are listed under Nomenclature.

It is taken that $t$ denotes time before a random breakdown has taken place during production uptime $t_1$, then the two different cases, presented in Sections 2.1 and 2.2, need to be studied.

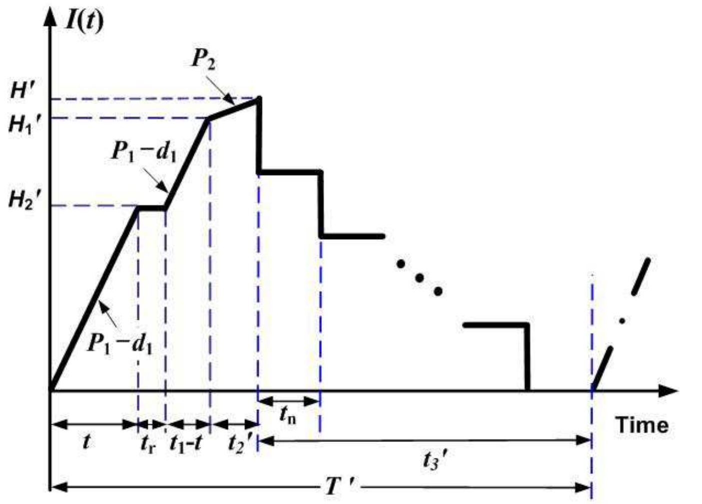


**Fig. 1** *On-hand inventory of perfect quality items in EPQ-based model with breakdown, rework, and multi-delivery policy*

### 2.1 CASE 1: A BREAKDOWN OCCURING DURING PRODUCTION UPTIME

In this case, time before a machine breakdown has taken place $t$, is smaller than the production uptime $t_1$. This means that a machine breakdown takes place during the production process. Under the AR policy, the production of an interrupted lot will be immediately resumed when the machine is fixed. The on-hand inventory level of perfect quality items at the time a random breakdown occurs is $H_2'$ (see Figure 1), and the level of on-hand inventory stays at $H_2'$ until the machine is fixed. $H_1'$ denotes the level of on-hand inventory in units when production of the remaining of interrupted lot is completed. Then, the rework process starts and the on-hand inventory level in the end of rework is $H'$. Fixed quantity of $n$ installments of a finished batch is delivered at a fixed interval of time during production downtime $t_3$. Figure 1 gives the cycle length $T'$ as:

$$T' = t + t_r + (t_1 - t) + t_2' + t_3' \tag{1}$$

The related costs in a production cycle for such a specific imperfect quality EPQ model with a random breakdown include: (1) a production setup cost per cycle; (2) a variable production cost; (3) a reworking cost; (4) a machine repairing cost; (5) fixed and the variable shipping costs; (6) an inventory holding cost during the rework process; (7) a holding cost for safety stocks (i.e., stocks to prevent shortage happening due to breakdown in the very early stage of the production cycle); and (8) an inventory holding cost during uptime, machine repairing time, and end items delivery period. Using the similar procedures from earlier studies [17], total production-inventory-delivery costs per cycle $TC_1(t_1)$ in the case of breakdown taking place (under AR policy) during uptime $t_1$, can be obtained as:

$$\begin{aligned} TC_1(t_1) &= K + C(P_1 t_1) + C_R(xP_1 t_1) + M + \\ &+ nK_1 + C_T(P_1 t_1) + h_1 \frac{d_1 t_1}{2} t_2' + h_3(\lambda t_r) T' + \\ &+ h\left[\frac{H_1' + d_1 t_1}{2} t_1 + \left(H_2' + d_1 t\right) t_r + \frac{H_1' + H'}{2}\left(t_2'\right) + \frac{n-1}{2n} H' t_3'\right] \end{aligned} \tag{2}$$

Since the proportion $x$ of defective items is assumed to be a random variable with a known probability density function, in order to take randomness of defective rate into account, one can use the expected values of $x$ in inventory cost analysis. Substituting all related system parameters and with further derivations, one obtains an expected production-inventory cost per cycle $E[TC_1(t_1)]$ for the proposed EPQ model as follows:

$$E\left[TC_1(t_1)\right] = K + M + \left[CP_1 + C_R P_1 E[x] + C_T P_1 + h_3 P_1 g\right]\cdot t_1 + \\ + nK_1 + htpg - t_1 \frac{hP_1 g}{2}\left(1 - \frac{1}{n}\right) + \\ + t_1^2 \left[\frac{hP_1^2 E[x]}{P_2}\left(1 - E[x]\right) + \frac{hP_1^2}{2\lambda}\left(1 - \frac{1}{n}\right) + \frac{hP_1^2 E[x]}{2P_2 n} + \frac{hP_1}{2n} + \frac{h_1 P_1^2 E[x]^2}{2P_2}\right] \tag{3}$$

## 2.2 CASE 2: NO BREAKDOWN OCCURING DURING PRODUCTION UPTIME

In this case, the time before a machine breakdown taking place $t$ is greater than, or equal to, the production uptime $t_1$. This means there is no breakdown taking place during the production process (see Figure 2).

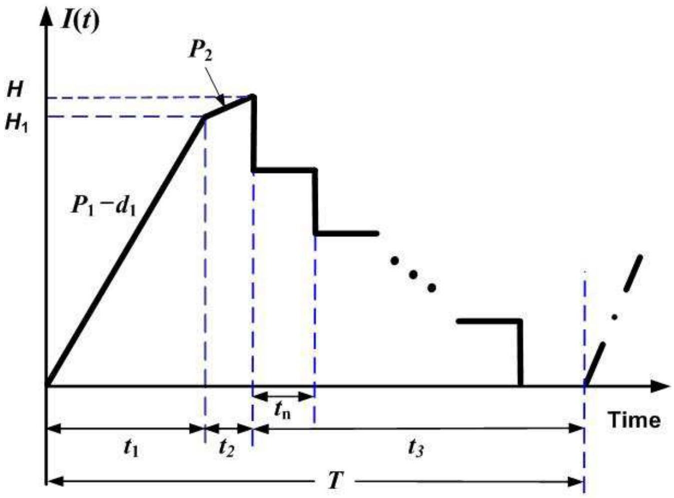


**Fig. 2** *On-hand inventory of perfect quality items in EPQ-based model with rework and multi-delivery policy*

As presented in Figure 2 the cycle length $T$ becomes:

$$T = t_1 + t_2 + t_3 = \frac{t_1 P_1}{\lambda} \tag{4}$$

Total inventory costs per cycle in case that breakdown does not occur, $TC_2(t_1)$ can be obtained as:

$$TC_2(t_1) = K + C(P_1 t_1) + C_R(xP_1 t_1) + nK_1 + C_T(P_1 t_1) + h_1 \frac{d_1 t_1}{2} t_2 + \\ + h_3(\lambda t_r)T + h\left[\frac{H_1 + d_1 t_1}{2} t_1 + \frac{H_1 + H}{2}(t_2) + \frac{n-1}{2n} H t_3\right] \tag{5}$$

Again, taking randomness of defective items into account, substituting all related parameters from Figure 2, and with further derivations, one can obtain the expected total production-inventory-delivery costs per cycle $E[TC_2(t_1)]$ as follows [29]:

$$E\left[TC_2(t_1)\right] = K + nK_1 + \left[CP_1 + C_R P_1 E[x] + C_T P_1 + h_3 P_1 g\right] t_1 + \\ + t_1^2 \left[\frac{hP_1^2 E[x]}{P_2}(1 - E[x]) + \frac{hP_1^2}{2\lambda}\left(1 - \frac{1}{n}\right) + \frac{hP_1^2 E[x]}{2P_2 n} + \frac{hP_1}{2n} + \frac{h_1 P_1^2 E[x]^2}{2P_2}\right] \tag{6}$$

## 2.3 INTEGRATION OF EPQ MODELS WITH/WITHOUT BREAKDOWN

Let $f(t)$ denote probability density function of random time $t$ before breakdown occurs. Also, let $F(t)$ be cumulative density function of $t$. Then, the long-run expected production-inventory-delivery costs per unit time (whether a breakdown takes place or not), $E[TCU(t_1)]$ is:

$$E\left[TCU(t_1)\right] = \frac{\left\{\int_0^{t_1} E\left[TC_1(t_1)\right] f(t)dt + \int_{t_1}^{\infty} E\left[TC_2(t_1)\right] f(t)dt\right\}}{E[\boldsymbol{T}]} \tag{7}$$

From Eqs (1) and (4), one obtains expected cycle length $E[\boldsymbol{T}]$ as:

$$E[\boldsymbol{T}] = \int_0^{t_1} E[T'] f(t)dt + \int_{t_1}^{\infty} E[T] f(t)dt = \frac{t_1 P_1}{\lambda} \tag{8}$$

Machine breakdown per unit time is assumed to be a random variable that follows the Poisson distribution with a mean that equals to $\beta$ per year. Therefore, the time to breakdown should obey exponential distribution with a density function $f(t)=\beta\, e^{-\beta t}$ and the cumulative density function $F(t)=1-e^{-\beta t}$.

Substituting $E[TC_1(t_1)]$, $E[TC_2(t_1)]$, and $E[\boldsymbol{T}]$ in Eq. (7), and solving the integration of mean time to breakdown in $E[TCU(t_1)]$ (see Appendix A), the long-run expected production-inventory-delivery cost per unit time can be obtained as:

$$E\left[TCU(t_1)\right] = \lambda \left\{ \begin{array}{l} \left[\frac{K}{t_1 P_1} + \frac{nK_1}{t_1 P_1}\right] + \alpha_3 + \left[\frac{M}{P_1} + \frac{hg}{\beta}\right]\left(\frac{1 - e^{-\beta t_1}}{t_1}\right) - \\ -hg\left(e^{-\beta t_1}\right) - \left[\frac{hg}{2} - \frac{hg}{2n}\right]\left(1 - e^{-\beta t_1}\right) + \frac{t_1 \cdot \omega}{2} \end{array} \right\} \tag{9}$$

where:

$$\alpha_3 = \left[C + C_R E[x] + C_T + h_3 g\right]$$

and:

$$\omega = \left[\frac{hP_1 E[x]}{P_2}(1 - E[x]) + \frac{hP_1}{\lambda}\left(1 - \frac{1}{n}\right) + \frac{h}{n} + \frac{hP_1 E[x]}{P_2 n} + \frac{h_1 P_1 E\left[x^2\right]}{P_2}\right]$$

## 3. CONVEXITY OF $E[TCU(t_1)]$

In order to determine the optimal production run time $t_1$*, one needs first to prove that $E[TCU(t_1)]$ is convex. Hence, Theorem 1 is proposed as follows: let $z(t_1)$ denote the following term:

$$z(t_1)=\frac{2\beta K+2K_1 n\beta+2\left[M\beta+hgP_1\right]\left(1-e^{-\beta t_1}\right)}{\left[P_1 t_1^2\frac{hg}{2}\left(1+\frac{1}{n}\right)+\left[M\beta+hgP_1\right]\left(2+\beta t_1\right)\right]\left(\beta^2 e^{-\beta t_1}\right)} \tag{10}$$

**Theorem 1: $E[TCU(t_1)]$ *is convex if* $0 < t_1 < z(t_1)$.**

If the second derivative of $E[TCU(t_1)]$ with respect to $t_1$, Eq. (11), is greater than zero, then $E[TCU(t_1)]$ is a convex function for all $t_1$ different from zero. Differentiating $E[TCU(t_1)]$ with respect to $t_1$ gives the second derivative as:

$$\frac{d^2E\left[TCU(t_1)\right]}{dt_1^2}=\lambda\left[\left[\frac{2K}{t_1^3 P_1}+\frac{2nK_1}{t_1^3 P_1}\right]+\left[hg-\frac{hg}{2\lambda}+\frac{hg}{2\lambda n}\right]\left(-\beta^2 e^{-\beta t_1}\right)+\right.$$
$$\left.+\left[\frac{M}{P_1}+\frac{hg}{\beta}\right]\left(\frac{2\left(1-e^{-\beta t_1}\right)}{t_1^3}-\frac{2\beta e^{-\beta t_1}}{t_1^2}-\frac{\beta^2 e^{-\beta t_1}}{t_1}\right)\right] \tag{11}$$

From Eq. (11), since an annual demand $\lambda$ is greater than zero, the following is obtained:

$$\text{if}\quad \lambda\left[\left[\frac{2K}{t_1^3 P_1}+\frac{2nK_1}{t_1^3 P_1}\right]+\left[hg-\frac{hg}{2\lambda}+\frac{hg}{2\lambda n}\right]\left(-\beta^2 e^{-\beta t_1}\right)+\right.$$
$$\left.+\left[\frac{M}{P_1}+\frac{hg}{\beta}\right]\left(\frac{2\left(1-e^{-\beta t_1}\right)}{t_1^3}-\frac{2\beta e^{-\beta t_1}}{t_1^2}-\frac{\beta^2 e^{-\beta t_1}}{t_1}\right)\right]>0\quad\text{then}\quad\frac{d^2E\left[TCU(t_1)\right]}{dt_1^2}>0 \tag{12}$$

With further arrangement, the following is obtained:

$$\frac{d^2E\left[TCU(t_1)\right]}{dt_1^2}>0\ \text{if}\ 0<t_1<\frac{2\beta K+2K_1 n\beta+2\left[M\beta+hgP_1\right]\left(1-e^{-\beta t_1}\right)}{\left[P_1 t_1^2\frac{hg}{2}\left(1+\frac{1}{n}\right)+\left[M\beta+hgP_1\right]\left(2+\beta t_1\right)\right]\left(\beta^2 e^{-\beta t_1}\right)}=z(t_1) \tag{13}$$

In order to minimize expected overall costs $E[TCU(t_1)]$, Eq. (13) must be satisfied. Now, to search for the optimal value of $t_1$* that yields a minimum cost, one can set first derivative of $E[TCU(t_1)]$ equal to $0$ as follows:

$$\frac{dE\left[TCU(t_1)\right]}{dt_1}=\lambda\left[\left[\frac{-K}{t_1^2 P_1}+\frac{-nK_1}{t_1^2 P_1}\right]+\left[hg-\frac{hg}{2}+\frac{hg}{2n}\right]\left(\beta e^{-\beta t_1}\right)+\right.$$
$$\left.+\left[\frac{M}{P_1}+\frac{hg}{\beta}\right]\left(\frac{-\left(1-e^{-\beta t_1}\right)}{t_1^2}+\frac{\beta e^{-\beta t_1}}{t_1}\right)+\frac{\omega}{2}\right]=0 \tag{14}$$

To find bounds for the optimal production run time, let:

$$t_{1U}^*=\sqrt{\frac{2\left(\gamma+K\beta+nK_1\beta\right)}{P_1\beta\omega}} \tag{15}$$

$$t_{1L}^{*}=\frac{-\gamma\pm\sqrt{\gamma^{2}+2P_{1}\left(2\beta\alpha_{4}+\omega\right)\left(K+nK_{1}\right)}}{P_{1}\left(2\beta\alpha_{4}+\omega\right)} \tag{16}$$

where $\gamma=\left(M\beta+hP_{1}g\right)$ and $\alpha_{4}=\frac{hg}{2}\left(1+\frac{1}{n}\right)$.

**Theorem 2: $t_{1L}^* < t_1^* < t_{1U}^*$**

To prove Theorem 2, it is necessary to first multiply both sides of Eq. (14) by $(2P_1t_1^2\beta)$:

$$\begin{aligned}&\left[P_{1}\beta^{2}hg\left(1+\frac{1}{n}\right)\left(e^{-\beta t_{1}}\right)+\left(P_{1}\beta\omega\right)\right]t_{1}^{2}+2\beta\left[M\beta+hP_{1}g\right]\left(e^{-\beta t_{1}}\right)t_{1}-\\&-2\left[\left(M\beta+hP_{1}g\right)\left(1-e^{-\beta t_{1}}\right)+\beta\left(K+nK_{1}\right)\right]=0\end{aligned} \tag{17}$$

or:

$$P_{1}\beta\left[2\beta\alpha_{4}\left(e^{-\beta t_{1}}\right)+\omega\right]t_{1}^{2}+\left[2\beta\gamma\left(e^{-\beta t_{1}}\right)\right]t_{1}-2\left[\gamma\left(1-e^{-\beta t_{1}}\right)+\beta\left(K+nK_{1}\right)\right]=0 \tag{18}$$

$$t_{1}^{*}=\text{the positive root of}\left\{\frac{-2\beta\gamma\left(e^{-\beta t_{1}}\right)\pm\sqrt{\left[2\beta\gamma\left(e^{-\beta t_{1}}\right)\right]^{2}-\left[4P_{1}\beta\left[2\beta\alpha_{4}\left(e^{-\beta t_{1}}\right)+\omega\right]\left[-2\left[\gamma\left(1-e^{-\beta t_{1}}\right)+\beta\left(K+nK_{1}\right)\right]\right]\right]}}{2P_{1}\beta\left[2\beta\alpha_{4}\left(e^{-\beta t_{1}}\right)+\omega\right]}\right\} \tag{19}$$

Now, in order to locate the optimal run time $t_1^*$, it is necessary to rearrange Eq. (18) as follows:

$$\left(e^{-\beta t_{1}}\right)2\left[\left(P_{1}\beta^{2}\alpha_{4}\right)t_{1}^{2}+\beta\gamma t_{1}+\gamma\right]=2\left[\gamma+\beta\left(K+nK_{1}\right)\right]-P_{1}\beta\omega t_{1}^{2} \tag{20}$$

$$e^{-\beta t_{1}}=\frac{2\left[\gamma+\beta\left(K+nK_{1}\right)\right]-P_{1}\beta\omega t_{1}^{2}}{2\left[\left(P_{1}\beta^{2}\alpha_{4}\right)t_{1}^{2}+\beta\gamma t_{1}+\gamma\right]} \tag{21}$$

Since $e^{-\beta t_1}$ is a complement of cumulative density function $F(t_1)=1-e^{-\beta t}$ and $0\le F(t_1)\le 1$, hence $0\le e^{-\beta t_1}\le 1$. Let $e^{-\beta t_1}=0$ and $e^{-\beta t_1}=1$ denote the bounds for $e^{-\beta t_1}$, respectively, from Eq. (19) we obtain:

$$t_{1U}^{*}=\sqrt{\frac{2\left(\gamma+K\beta+nK_{1}\beta\right)}{P_{1}\beta\omega}}$$

$$t_{1L}^{*}=\text{the positive root of}\left\{\frac{-\gamma\pm\sqrt{\gamma^{2}+2P_{1}\left(2\beta\alpha_{4}+\omega\right)\left(K+nK_{1}\right)}}{P_{1}\left(2\beta\alpha_{4}+\omega\right)}\right\}$$

and $t_{1L}^* < t_1^* < t_{1U}^*$.

Although the optimal run time $t_1^*$ cannot be expressed in a closed form, it can be located through the use of a proposed recursive searching algorithm (see Appendix B), based on the existence of bounds for $e^{-\beta t_1}$ and $t_1^*$.

## 4. NUMERICAL EXAMPLE

It was assumed that a manufactured product can be produced at a rate of *10000* units per year, and that it experiences a relatively flat demand of *4000* units per year. During its production uptime, a random defective rate is assumed and it follows a uniform distribution over the interval [*0*, *0.2*]. All defective items can be repaired through rework process at a rate $P_2$=*5000* units per year in the end of production.

Furthermore, the machine in production system is subject to a random breakdown that follows a Poisson distribution with mean $\beta$=*0.5* times per year. An AR policy is used when a random breakdown takes place. Other parameters of this example are given as follows: $K$=*\$450* for each production run; $h$=*\$0.*6 per item per unit time; $C$=*\$2* per item; $C_R$=*\$0.5* repaired cost for each item reworked; $h_1$=*\$0.8* per item reworked per unit time; $M$=*\$500* repair cost for each breakdown; $g$=*0.018* years (time needed to repair and restore the machine); $n$=*4* installments of finished batch are delivered per cycle; $K_1$=*\$80* per shipment, a fixed cost; and $C_T$=*\$0.001* per item delivered.

For the convexity of $E[TCU(t_1)]$ (Eq. (13)), using both upper and lower bounds of $t_1$* in Eq. (13), one finds out that it holds. Applying Eqs. (16) and (9) one obtains $t_{1L}$*=*0.30352* (years) and $E[TCU(t_{1L}*)]$=*\$10,222.89*. Then, applying Eqs. (15) and (9) one has $t_{1U}$*=*0.45605* (years) and $E[TCU(t_{1U}*)]$=*\$10,837.76*.

Further, as stated earlier, since $E[TCU(t_1)]$ is convex and the optimal run time $t_1$* falls within the interval of [$t_{1L}$*, $t_{1U}$*] (see Theorems 1 & 2), using a proposed recursive searching algorithm (see Appendix B), one can locate the optimal run time $t_1$*. Step-by-step iterations and their results are displayed in Table 1. When $\beta$=*0.5*, the optimal run time $t_1$*=*0.3295* years and the optimal expected costs per unit time $E[TCU(t_1*)]$=*\$10,216.59* (see Figure 3).

**Table 1** *Iterations of the proposed recursive searching algorithm for locating the optimal run time $t_1$**

| $\beta$ | Iteration | $y_L = e^{-\beta t_{1U}}$ | $t_{1U}$* | $y_U = e^{-\beta t_{1L}}$ | $t_{1L}$* | Difference between $t_{1U}$* & $t_{1L}$* | [U] $E[TCU(t_{1U}*)]$ | [L] $E[TCU(t_{1L}*)]$ | Difference between [U] and [L] |
|---|---|---|---|---|---|---|---|---|---|
| 0.5 | initial | 0.00000 | 0.45605 | 1.00000 | 0.30352 | 0.15253 | \$10,837.76 | \$10,222.89 | \$614.87 |
| | 2nd | 0.79611 | 0.33806 | 0.85919 | 0.32762 | 0.01044 | \$10,259.50 | \$10,216.62 | \$42.88 |
| | 3rd | 0.84448 | 0.33008 | 0.84890 | 0.32934 | 0.00074 | \$10,219.61 | \$10,216.59 | \$3.02 |
| | 4th | 0.84786 | 0.32952 | 0.84817 | 0.32946 | 0.00005 | \$10,216.80 | \$10,216.59 | \$0.21 |
| | 5th | 0.84810 | 0.32948 | 0.84812 | 0.32947 | 0.00000 | \$10,216.60 | \$10,216.59 | \$0.01 |
| | **6th** | **0.84812** | **0.32947** | **0.84812** | **0.32947** | **0.00000** | **\$10,216.59** | **\$10,216.59** | **\$0.00** |

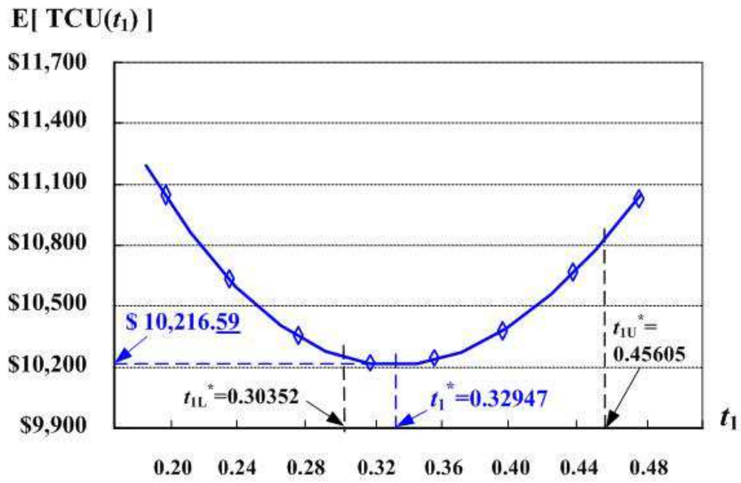


**Fig. 3** *The behavior of E[TCU($t_1$)] with respect to production run time $t_1$*

The behavior of $E[TCU(t_1)]$ with respect to $x$ and $1/\beta$ is illustrated in Figure 4. One notes that as the mean time between breakdowns $1/\beta$ decreases, the value of $E[TCU(t_1*)]$ increases. Also, as $x$ increases, $E[TCU(t_1*)]$ goes up significantly, too.

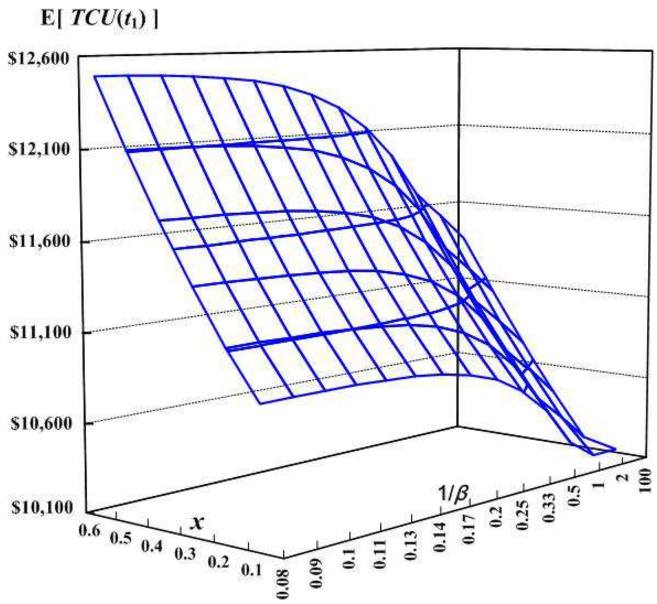


**Fig. 4** *The behavior of E[TCU($t_1$)] with respect to x and 1/β*

## 5. CONCLUDING REMARKS

An EPQ-based inventory model with a random breakdown, rework, and multi-delivery policy is studied. An AR inventory control policy is adopted when a machine breakdown occurs. We present a complete solution procedure which includes mathematical modelling, derivations of cost functions for the proposed EPQ models with and without breakdown, integration of cost functions, theorems on convexity and on bounds of the production run time, development of a recursive run time searching algorithm, and a numerical demonstration aiming at confirmation of the entire solution procedure. Without an in-depth study on such a specific imperfect production system with breakdown, multi-delivery, and rework, the optimal run time and related facts of the system cannot be revealed. For future research, it would be interesting to study the effects of variable production rates on the same model.

## 6. ACKNOWLEDGEMENT

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## 7. APPENDIX – A

**Nomenclature**

$t_1$ = optimal production time (i.e. uptime) to be determined for the proposed EPQ model,

$t$ = production time before a random breakdown occurs,

$\beta$ = number of breakdowns per year, a random variable that follows the Poisson distribution,

$t_r$ = time required for repairing the machine,

$t_2'$ = time needed for the rework of defective items when machine breakdown takes place,

$t_3'$ = time needed for consuming all available perfect quality items when breakdown takes place,

$H_2'$= the level of on-hand inventory in units when random breakdown takes place,

$H_1'$= the level of on-hand inventory in units when a regular production ends (in the case of a breakdown),

$H'$ = the maximum level of on-hand inventory in units when a regular production ends in the EPQ model with random breakdown,

$Q$ = production lot size per cycle,

$T'$ = cycle length in the case of machine breakdown,

$h_3$ = unit holding cost for safety stock,

$TC_1(t_1)$ = the total production-inventory-delivery costs per cycle in the case of breakdown,

$E[TC_1(t_1)]$ = the expected production-inventory-delivery costs per cycle in the case of breakdown,

$t_2$ = time required for the rework of defective items in case a breakdown does not occur,

$t_3$ = time required for depleting all available perfect quality items in case a breakdown does not occur,

$H_1$ = the level of on-hand inventory in units when a regular production ends in the EPQ model without any occurrence of a machine failure,

$H$ = the maximum level of on-hand inventory in units when the rework process ends in the EPQ model without any occurrence of a machine breakdown,

$T$ = cycle length in case the breakdown does not occur,

$I(t)$ = the level of on-hand inventory of perfect quality items at a time t,

$TC_2(t_1)$ = total production-inventory-delivery costs per cycle in case a breakdown does not occur,

$E[TC_2(t_1)]$ = the expected production-inventory-delivery costs per cycle in the case of no breakdown occurrence during uptime,

$\boldsymbol{T}$ = cycle length regardless of whether a machine breakdown takes place or not,

$TCU(t_1)$ = the total production-inventory-delivery costs per unit time regardless of whether a breakdown takes place or not,

$E[TCU(t_1)]$ = the long-run expected production-inventory-delivery costs per unit time regardless of whether a breakdown takes place or not.

## 8. APPENDIX – B

A proposed recursive searching algorithm for finding $t_1$*:

Although the optimal run time $t_1$* cannot be expressed in a closed form, it can be located through the use of following searching algorithm based on the existence of bounds for $e^{-\beta t_1}$ and $t_1$*. Recall Eq. (21):

$$e^{-\beta t_1} = \frac{2\left[\gamma+\beta\left(K+nK_1\right)\right]-P_1\beta\omega t_1^2}{2\left[\left(P_1\beta^2\alpha_4\right)t_1^2+\beta\gamma t_1+\gamma\right]}$$

Because $e^{-\beta t_1}$ is complement of cumulative density function, therefore, $0 \le e^{-\beta t_1} \le 1$.

Let:

$$y(t_1)=e^{-\beta t_1} = \frac{2\left[\gamma+\beta\left(K+nK_1\right)\right]-P_1\beta\omega t_1^2}{2\left[\left(P_1\beta^2\alpha_4\right)t_1^2+\beta\gamma t_1+\gamma\right]} \quad 0\le y(t_1)\le 1$$

The following recursive searching techniques to find $t_1$* are proposed in this study:

(1) Let $y(t_1)$=0 and $y(t_1)$=1 initially and compute the upper and lower bounds for $t_1^*$, respectively (i.e., the initial values of [$t_{1L}^*$, $t_{1U}^*$]).

(2) Substitute the current values of [$t_{1L}^*$, $t_{1U}^*$] in $e^{-\beta t_1}$ and calculate the new bounds (denoted as $y_L$ and $y_U$) for $e^{-\beta t_1}$. Hence, $y_L < z(t_1) < y_U$.

(3) Let $z(t_1)=y_L$ and $z(t_1)=y_U$ and compute the new upper and lower bounds for $t_1^*$, respectively (i.e. to update the current values of [$t_{1L}^*$, $t_{1U}^*$]).

(4) Repeat steps 2 and 3, until there is no significant difference between $t_{1L}^*$ and $t_{1U}^*$ (or there is no significant difference in terms of their effects on $E[TCU(t_1*)]$).

(5) Stop. The optimal production run time $t_1^*$ is obtained.

A step-by-step demonstration of recursive searching algorithm is presented in Table 1.

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## PRIKAZ EPQ MODELA SA SLUČAJNIM KVAROVIMA, DORADOM I DISKONTINUIRANOM ISPORUKOM

*Svrha ovog rada jest određivanje optimalnog vremena proizvodnje za model ekonomične količine proizvodnje sa Poisson-ovom razdiobom kvarova, popravkom nesukladnih proizvedenih jedinica, te višekratnom isporukom konačnog proizvoda. Pretpostavlja se da se nesukladne proizvedene jedinice mogu slučajno proizvesti i naknadno popraviti pomoću procesa dorade na kraju proizvodnog procesa. Kvar opreme za proizvodnju smatra se slučajnom veličinom koja se pojavljuje po Poisson-ovoj razdiobi. Kada se dogodi kvar, stroj se odmah popravlja, te se pokreće u modu otkazivanje/nastavak. U ovom modu, proizvodnja prekinute serije proizvoda se trenutačno nastavlja nakon popravka stroja. Nakon završetka procesa dorade, fiksan broj od n proizvoda završene proizvodne serije isporučuje se kupcu u fiksnom vremenskom intervalu. U ovom je radu korišteno matematičko modeliranje u kombinaciji s numeričkom analizom. Predloženi su i najzad dokazani teoremi konveksnosti za očekivanu funkciju troška sustava i granično vrijeme proizvodnje. Razvijen je rekurzivni algoritam za iznalaženje optimalnog vremena proizvodnje unutar definiranih granica. Na kraju rada prikazan je numerički primjer radi praktične demonstracije rezultata opisanog istraživanja.*

**KLJUČNE RIJEČI:** *model ekonomične količine proizvodnje, kvar, višestruka isporuka, dorada, mod otkazivanje/nastavak, optimizacija.*